Selective Averaging with Application to Phase Reduction

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Abstract—For a class of vector fields, we show that one can selectively average terms which are of the same order in a small parameter, giving an extension of standard averaging results. Such selective averaging is illustrated for the phase reduction of a system of oscillators with both coupling and external input, for which the coupling can be averaged to give a term which only depends on phase differences, while the external input term is not averaged.

1. Introduction

We will consider vector fields of the form

$$\dot{x} = \varepsilon f^0(x,t) + \varepsilon f^1(x,t), \qquad x(0) = x_0, \tag{1}$$

where $f^i : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ is continuous in x and t for $i = 0, 1$. We assume that

$$M_i \equiv \sup_{x\in D} \sup_{0\le\varepsilon t\le L} \|f^i(x,t)\| < \infty, \qquad i = 0, 1,$$

where $D \subset \mathbb{R}^n$ and L is chosen so that $x(t) \in D$ for all $0 \le t \le L/\varepsilon$. Moreover, we assume that f^0 and f^1 satisfy

$$\|f^i(x,t) - f^i(y,t)\| \le \lambda_{f^i}\|x - y\|, \qquad i = 0, 1$$

for $x, y \in D$; here λ_{f^i} is called a Lipschitz constant for f^i.

In this paper, we will extend averaging theorems from [7] to the case that we call *selective averaging*, in which certain terms for a vector field are averaged while others are not. In (1), this corresponds to averaging f^0 but not f^1. The use of selective averaging will be illustrated for the phase reduction of a system of oscillators with both coupling and external input.

2. The Selective Averaging Theorem

We first define the *local average* f_T of a function $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ to be

$$f_T(x,t) := \frac{1}{T}\int_0^T f(x, t+s)\mathrm{d}s.$$

The following lemmas will allow us to prove the Selective Averaging Theorem.

Lemma 2.1 (Lemma 4.2.3 from [7]) *If the continuous vector field $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ is T-periodic in t, then*

$$f_T(x,t) = \overline{f}(x) = \frac{1}{T}\int_0^T f(x,s)ds.$$

Lemma 2.2 *Consider the initial value problem (1). With t on the time scale $\frac{1}{\varepsilon}$, the solution $x(t)$ satisfies*

$$\Bigg\| x_T(t) - x_0 - \varepsilon\int_0^t f_T^0(x(\sigma),\sigma)\mathrm{d}\sigma - \frac{\varepsilon}{T}\int_0^T\int_0^{t+s} f^1(x(\sigma),\sigma)\mathrm{d}\sigma\mathrm{d}s \Bigg\| \le \frac{1}{2}\varepsilon T((1+\lambda_{f^0}L)M_0 + \lambda_{f^0}LM_1).$$

Proof The proof is similar to [7, Lemma 4.2.7]. We express the solution to (1) as

$$x(t) = x_0 + \varepsilon\int_0^t f^0(x(\sigma),\sigma)\mathrm{d}\sigma + \varepsilon\int_0^t f^1(x(\sigma),\sigma)\mathrm{d}\sigma.$$

Calculating the local average of the solution, we obtain

$$x_T(t) = x_0 + \frac{\varepsilon}{T}\int_0^T\int_0^{t+s} f^0(x(\sigma),\sigma)\mathrm{d}\sigma\mathrm{d}s + \frac{\varepsilon}{T}\int_0^T\int_0^{t+s} f^1(x(\sigma),\sigma)\mathrm{d}\sigma\mathrm{d}s.$$

Now, the term

$$\begin{aligned}&\frac{\varepsilon}{T}\int_0^T\int_0^{t+s} f^0(x(\sigma),\sigma)\mathrm{d}\sigma\mathrm{d}s\\ &= \frac{\varepsilon}{T}\int_0^T\int_0^{t} f^0(x(\sigma+s),\sigma+s)\mathrm{d}\sigma\mathrm{d}s + \varepsilon R_1\\ &= \frac{\varepsilon}{T}\int_0^t\int_0^{T} f^0(x(\sigma),\sigma+s)\mathrm{d}s\mathrm{d}\sigma + \varepsilon R_1 + \varepsilon R_2,\end{aligned}$$

where

$$\|R_1\| = \left\| \frac{1}{T}\int_0^T\int_0^s f^0(x(\sigma),\sigma)\mathrm{d}\sigma\mathrm{d}s \right\| \le \frac{1}{T}\int_0^T\int_0^s M_0\mathrm{d}\sigma\mathrm{d}s = \frac{1}{2}M_0T,$$

and

$$\|R_2\| = \Bigg\| \frac{1}{T}\int_0^t\int_0^T [f^0(x(\sigma+s),\sigma+s) - f^0(x(\sigma),\sigma+s)]\mathrm{d}s\mathrm{d}\sigma \Bigg\|$$

$$\le \frac{\lambda_{f^0}}{T}\int_0^t\int_0^T \|x(\sigma+s) - x(\sigma)\|\mathrm{d}s\mathrm{d}\sigma$$

$$= \varepsilon\frac{\lambda_{f^0}}{T}\int_0^t\int_0^T\left\|\int_\sigma^{\sigma+s}(f^0(x(\zeta),\zeta)+f^1(x(\zeta),\zeta))\mathrm{d}\zeta\right\|\mathrm{d}s\mathrm{d}\sigma$$

$$\leq \varepsilon\frac{\lambda_{f^0}}{T}\int_0^t\int_0^T\int_\sigma^{\sigma+s}\|f^0(x(\zeta),\zeta)+f^1(x(\zeta),\zeta)\|\mathrm{d}\zeta\mathrm{d}s\mathrm{d}\sigma$$

$$\leq \varepsilon\frac{\lambda_{f^0}}{T}\int_0^t\int_0^T\int_\sigma^{\sigma+s}(\|f^0(x(\zeta),\zeta)\|+\|f^1(x(\zeta),\zeta)\|)\mathrm{d}\zeta\mathrm{d}s\mathrm{d}\sigma$$

$$\leq \varepsilon\frac{\lambda_{f^0}}{T}\int_0^t\int_0^T(M_0s+M_1s)\mathrm{d}s\mathrm{d}\sigma = \frac{1}{2}\varepsilon\lambda_{f^0}t(M_0+M_1)T$$

$$\leq \frac{1}{2}\lambda_{f^0}L(M_0+M_1)T.$$

Putting these expressions together gives the result. ∎

Lemma 2.3 *Consider the initial value problem (1). If y is the solution of the initial value problem*

$$\dot{y} = \varepsilon f_T^0(y,t)+\varepsilon f^1(y,t), \qquad y(0)=x_0,$$

then $x(t) = y(t) + O(\varepsilon T)$ *on the time scale* $\frac{1}{\varepsilon}$.

Proof The proof is similar to [7, Lemma 4.2.8].

$$\left\|x(t)-x_0-\varepsilon\int_0^t f_T^0(x(\sigma),\sigma)\mathrm{d}\sigma-\varepsilon\int_0^t f^1(x(\sigma),\sigma)\mathrm{d}\sigma\right\|$$

$$\leq \left\|x(t)-x_0-\varepsilon\int_0^t f_T^0(x(\sigma),\sigma)\mathrm{d}s - \frac{\varepsilon}{T}\int_0^T\int_0^{t+s}f^1(x(\sigma),\sigma)\mathrm{d}\sigma\mathrm{d}s\right\|$$

$$+\left\|\frac{\varepsilon}{T}\int_0^T\int_0^{t+s}f^1(x(\sigma),\sigma)\mathrm{d}\sigma\mathrm{d}s-\varepsilon\int_0^t f^1(x(\sigma),\sigma)\mathrm{d}\sigma\right\|.$$

Now, the term

$$\left\|\frac{\varepsilon}{T}\int_0^T\int_0^{t+s}f^1(x(\sigma),\sigma)\mathrm{d}\sigma\mathrm{d}s-\varepsilon\int_0^t f^1(x(\sigma),\sigma)\mathrm{d}\sigma\right\|$$

$$= \left\|\frac{\varepsilon}{T}\int_0^T\int_0^{t+s}f^1(x(\sigma),\sigma)\mathrm{d}\sigma\mathrm{d}s - \frac{\varepsilon}{T}\int_0^T\int_0^t f^1(x(\sigma),\sigma)\mathrm{d}\sigma\mathrm{d}s\right\|$$

$$= \left\|\frac{\varepsilon}{T}\int_0^T\left(\int_0^{t+s}f^1(x(\sigma),\sigma)\mathrm{d}\sigma-\int_0^t f^1(x(\sigma),\sigma)\mathrm{d}\sigma\right)\mathrm{d}s\right\|$$

$$= \left\|\frac{\varepsilon}{T}\int_0^T\left(\int_t^{t+s}f^1(x(\sigma),\sigma)\mathrm{d}\sigma\right)\mathrm{d}s\right\|$$

$$\leq \left\|\frac{\varepsilon}{T}\int_0^T sM_1\mathrm{d}s\right\| = \frac{1}{2}\varepsilon M_1T.$$

Putting this together with Lemma 2.2, we obtain

$$\left\|x(t)-x_0-\varepsilon\int_0^t f_T^0(x(\sigma),\sigma)\mathrm{d}\sigma-\varepsilon\int_0^t f^1(x(\sigma),\sigma)\mathrm{d}\sigma\right\|$$

$$\leq \frac{1}{2}\varepsilon T((1+\lambda_{f^0}L)M_0+\lambda_{f^0}LM_1)+\frac{1}{2}\varepsilon M_1T.$$

Thus, we have

$$x(t) = x_0+\varepsilon\int_0^t f_T^0(x(\sigma),\sigma)\mathrm{d}\sigma+\varepsilon\int_0^t f^1(x(\sigma),\sigma)\mathrm{d}\sigma + O(\varepsilon T).$$

Now,

$$y(t) = x_0+\varepsilon\int_0^t f_T^0(y(\sigma),\sigma)\mathrm{d}\sigma+\varepsilon\int_0^t f^1(y(\sigma),\sigma)\mathrm{d}\sigma,$$

so

$$x(t)-y(t) = \varepsilon\int_0^t[f_T^0(x(\sigma),\sigma)-f_T^0(y(\sigma),\sigma)]\mathrm{d}\sigma$$

$$+\varepsilon\int_0^t[f^1(x(\sigma),\sigma)-f^1(y(\sigma),\sigma)]\mathrm{d}\sigma+O(\varepsilon t).$$

Therefore,

$$\|x(t)-y(t)\| \leq \varepsilon\int_0^t(\lambda_{f^0}+\lambda_{f^1})\|x(\sigma)-y(\sigma)\|\mathrm{d}\sigma+O(\varepsilon T).$$

Then, applying Gronwall's Lemma [7, Lemma 1.3.1],

$$\|x(t)-y(t)\| = O\left(\varepsilon T e^{\varepsilon(\lambda_{f^0}+\lambda_{f^1})t}\right).$$ ∎

We can now prove the following.

Theorem 2.4 **(Selective Averaging Theorem)** *Let* $x(t)$ *be the solution to*

$$\dot{x} = \varepsilon f^0(x,t)+\varepsilon f^1(x,t), \qquad x(0)=x_0, \tag{2}$$

and let $y(t)$ *be the solution to*

$$\dot{y} = \varepsilon\overline{f^0}(y)+\varepsilon f^1(y,t), \qquad y(0)=x_0, \tag{3}$$

where f^0 *is* T*-periodic, and* f^0 *and* f^1 *satisfy the assumptions given in Section 1. Then*

$$\|x(t)-y(t)\| = O(\varepsilon)$$

on the time scale $1/\varepsilon$.

Proof This follows from Lemmas 2.1 and 2.3. ∎

3. Application to Phase Reduction

A powerful technique for analyzing biological oscillators is the rigorous reduction to a phase model, with a single variable for each oscillator describing the phase of the oscillation with respect to some reference state [5, 9, 3]. This tremendous reduction in the dimensionality and complexity of a system often retains enough information to yield a useful understanding of its dynamics, and can allow for the implementation of phase-based control algorithms.

Phase reduction is commonly applied to systems of coupled oscillators, where in the limit of weak coupling one can use averaging to obtain terms which only depend on the phase differences of the oscillators; see, for example, [2]. Phase reduction has also been applied to systems of uncoupled oscillators which receive an external input, for example in [1]. Here we consider phase reduction for coupled oscillators with an external input; by averaging only the coupling term, we provide justification for models that are sometimes useful for neural control problems, e.g., [8, 6].

Suppose that the system

$$\frac{d\mathbf{x}}{dt} = \mathbf{F}(\mathbf{x})$$

has a periodic orbit $\mathbf{x}_\gamma(t)$ with period $T = \frac{2\pi}{\omega}$. Now consider

$$\frac{d\mathbf{x}_i}{dt} = \mathbf{F}(\mathbf{x}_i) + \varepsilon \sum_j \mathbf{p}(\mathbf{x}_i, \mathbf{x}_j) + \varepsilon u(t)\hat{e}_1, \qquad i = 1, \cdots, N,$$

where $\mathbf{x}_i$ is the state of the i^{th} oscillator, $\mathbf{p}$ represents coupling between oscillators, $u(t)$ is the external input, and $\hat{e}_1$ is a unit vector in the x_1-direction. (For a neuron, this could correspond to an input $u(t)$ in the voltage equation.) Here, for simplicity we have assumed that all oscillators are identical, have identical coupling to all other oscillators, and receive the same input $u(t)$. We transform to phase variables as follows, cf. [5]:

$$\begin{aligned}\frac{d\theta_i}{dt} &= \frac{\partial\theta_i}{\partial\mathbf{x}_i} \cdot \frac{d\mathbf{x}_i}{dt} = \frac{\partial\theta_i}{\partial\mathbf{x}_i} \cdot \left(\mathbf{F}(\mathbf{x}_i) + \varepsilon \sum_j \mathbf{p}(\mathbf{x}_i, \mathbf{x}_j) + \varepsilon u(t)\hat{e}_1\right) \\ &= \omega + \varepsilon \frac{\partial\theta_i}{\partial\mathbf{x}_i} \cdot \sum_j \mathbf{p}(\mathbf{x}_i, \mathbf{x}_j) + \varepsilon \frac{\partial\theta_i}{\partial\mathbf{x}_i} \cdot (u(t)\hat{e}_1).\end{aligned}$$

To lowest order in ε,

$$\frac{d\theta_i}{dt} = \omega + \varepsilon\mathbf{Z}(\theta_i) \cdot \sum_j \mathbf{p}(\theta_i, \theta_j) + \varepsilon\mathbf{Z}(\theta_i) \cdot (u(t)\hat{e}_1),$$

$$\mathbf{Z}(\theta_i) = \left.\frac{\partial\theta_i}{\partial\mathbf{x}_i}\right|_{\mathbf{x}_\gamma(\theta_i)}, \qquad \mathbf{p}(\theta_i, \theta_j) = \mathbf{p}(\mathbf{x}_\gamma(\theta_i), \mathbf{x}_\gamma(\theta_j)).$$

Here, $\mathbf{Z}(\theta)$ is known as the phase response curve [9]. Let $\theta_i = \phi_i + \omega t$:

$$\frac{d\phi_i}{dt} = \varepsilon\mathbf{Z}(\phi_i+\omega t) \cdot \sum_j \mathbf{p}(\phi_i+\omega t, \phi_j+\omega t) + \varepsilon\mathbf{Z}(\phi_i+\omega t) \cdot (u(t)\hat{e}_1).$$

Now, apply the Selective Averaging Theorem to average the coupling term (to use this theorem, we can consider the lift of ϕ_i to $\mathbb{R}$):

$$\frac{d\varphi_i}{dt} = \frac{\varepsilon}{T} \int_0^T \mathbf{Z}(\varphi_i + \omega t) \cdot \sum_j \mathbf{p}(\varphi_i + \omega t, \underbrace{\varphi_j + \omega t}_{\varphi_j - \varphi_i + \varphi_i + \omega t})dt$$

$$+\varepsilon\mathbf{Z}(\varphi_i + \omega t) \cdot (u(t)\hat{e}_1).$$

Let $s = \varphi_i + \omega t$, which gives

$$\frac{d\varphi_i}{dt} = \frac{\varepsilon}{2\pi} \sum_j \int_0^{2\pi} \mathbf{Z}(s) \cdot \mathbf{p}(s, \varphi_j - \varphi_i + s)ds$$

$$+\varepsilon\mathbf{Z}(\varphi_i + \omega t) \cdot (u(t)\hat{e}_1).$$

Then, letting $\vartheta_i = \varphi_i + \omega t$,

$$\frac{d\vartheta_i}{dt} = \omega + \frac{\varepsilon}{2\pi} \sum_j \int_0^{2\pi} \mathbf{Z}(s) \cdot \mathbf{p}(s, \vartheta_j - \vartheta_i + s)ds.$$

$$+\varepsilon\mathbf{Z}(\vartheta_i) \cdot (u(t)\hat{e}_1)$$

That is,

$$\frac{d\vartheta_i}{dt} = \omega + \varepsilon \sum_j h(\vartheta_j - \vartheta_i) + \varepsilon\mathbf{Z}(\vartheta_i) \cdot (u(t)\hat{e}_1),$$

where

$$h(\psi) = \frac{1}{2\pi} \int_0^{2\pi} \mathbf{Z}(s) \cdot \mathbf{p}(s, \psi + s)ds.$$

From the Selective Averaging Theorem, we expect $\theta_i(t) - \vartheta_i(t) = \phi_i(t) - \varphi_i(t) = O(\varepsilon)$ on the time scale $1/\varepsilon$.

We note that a similar phase reduction result has been obtained using different means in [4].

4. The General Selective Averaging Theorem

Following [7], we can also prove a general selective averaging theorem. Consider a vector field $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ which is continuous in x and t, and Lipschitz continuous in x on $D \subset \mathbb{R}^n$. If the average

$$\overline{f}(x) = \lim_{T\to\infty} \frac{1}{T} \int_0^T f(x, s)\mathrm{d}s$$

exists and the limit is uniform in x on compact sets $K \subset D$, then we call f a *Krylov-Bogoliubov-Mitropolsky (KBM) vector field*. The following lemmas will allow us to prove the General Selective Averaging Theorem.

Lemma 4.1 (Lemma 4.3.1 from [7]) *If f^0 is a KBM vector field, and assuming that $\varepsilon T = o(1)$ as $\varepsilon \downarrow 0$ (that is, $\lim_{\varepsilon\downarrow 0} \varepsilon T = 0$), then on a time scale $\frac{1}{\varepsilon}$ one has*

$$f_T^0(x, t) = \overline{f^0}(x) + O\left(\frac{\delta_0(\varepsilon)}{\varepsilon T}\right)$$

where

$$\delta_0(\varepsilon) = \sup_{x\in D} \sup_{t\in[0,\frac{L}{\varepsilon})} \varepsilon \left\| \int_0^t \left[f^0(x, s) - \overline{f^0}(x)\right] \mathrm{d}s \right\|.$$

Lemma 4.2 *Let y be the solution of the initial value problem*

$$\dot{y} = \varepsilon f_T^0(y, t) + \varepsilon f^1(y, t), \qquad y(0) = x_0, \tag{4}$$

and suppose f^0 is a KBM vector field with order function $\delta_0(\varepsilon)$. Then the solution of

$$\dot{z} = \varepsilon\overline{f^0}(z) + \varepsilon f^1(z, t), \qquad z(0) = x_0 \tag{5}$$

satisfies

$$y(t) = z(t) + O\left(\frac{\delta_0(\varepsilon)}{\varepsilon T}\right)$$

with t on a time scale $\frac{1}{\varepsilon}$.

Proof The proof is similar to ([7, Lemma 4.3.5]). We express the solutions to (4) and (5) respectively as:

$$y(t) = x_0 + \varepsilon \int_0^t f_T^0(y(s), s)ds + \varepsilon \int_0^t f^1(y(s), s)ds,$$

$$z(t) = x_0 + \varepsilon \int_0^t \overline{f^0}(z(s))ds + \varepsilon \int_0^t f^1(z(s), s)ds.$$

Using Lemma 4.1,

$$y(t) - z(t) = \varepsilon \int_0^t \left(\overline{f^0}(y(s)) - \overline{f^0}(z(s))\right) \mathrm{d}s + O\left(\frac{\delta_0(\varepsilon)t}{T}\right)$$
$$+\varepsilon \int_0^t (f^1(y(s), s) - f^1(z(s), s))\mathrm{d}s.$$

Since it can be shown that

$$\left\| \int_0^t \left(\overline{f^0}(y(s)) - \overline{f^0}(z(s))\right) ds \right\| \le \int_0^t \lambda_{f^0} \|y(s) - z(s)\| ds,$$

we find that

$$\|y(t) - z(t)\| \le \varepsilon \int_0^t (\lambda_{f^0} + \lambda_{f^1})\|y(s) - z(s)\| ds$$
$$+O\left(\frac{\delta_0(\varepsilon)t}{T}\right).$$

Applying Gronwall's Lemma [7, Lemma 1.3.1] we obtain

$$\|y(t) - z(t)\| = O\left(\frac{\delta_0(\varepsilon)t}{T} e^{\varepsilon(\lambda_{f^0} + \lambda_{f^1})t}\right).$$

The result follows by taking t on the time scale $1/\varepsilon$. ∎

We can now prove the following.

Theorem 4.3 (General Selective Averaging Theorem) *Let x be a solution of the initial value problem*

$$\dot{x} = \varepsilon f^0(x, t) + \varepsilon f^1(x, t), \qquad x(0) = x_0.$$

We assume that f^0 *is a KBM-vector field with order function* $\delta_0(\varepsilon)$. *Let z be the solution of the initial value problem*

$$\dot{z} = \varepsilon \overline{f^0}(z) + \varepsilon f^1(z, t), \qquad z(0) = x_0.$$

Then

$$x(t) = z(t) + O(\sqrt{\delta_0(\epsilon)}).$$

Proof The proof similar to [7, Theorem 4.3.6]. By Lemma 2.3 we know that the solution y of

$$\dot{y} = \varepsilon f_T^0(y, t) + \varepsilon f^1(y, t)$$

satisfies

$$x(t) = y(t) + O(\varepsilon T)$$

on a time scale $\frac{1}{\varepsilon}$. Also, from Lemma 4.2,

$$y(t) = z(t) + O\left(\frac{\delta_0(\varepsilon)}{\varepsilon T}\right).$$

Then, from the triangle inequality

$$\|x(t) - z(t)\| \le \|x(t) - y(t)\| + \|y(t) - z(t)\|,$$

we have

$$x(t) = z(t) + O(\varepsilon T) + O\left(\frac{\delta_0(\varepsilon)}{\varepsilon T}\right).$$

If we let $T = \sqrt{\delta_0(\varepsilon)}/\varepsilon$, then the result follows. ∎

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