Forced Symmetry Breaking as a Mechanism for Bursting

J. Moehlis and E. Knobloch
Department of Physics, University of California, Berkeley, California 94720
(Received 16 December 1997)

A robust mechanism generating regular or irregular bursts of large dynamic range near threshold is described. The mechanism arises in the interaction between oscillatory modes of odd and even parity in systems of large but finite aspect ratio, and provides an explanation for the bursting behavior observed in binary fluid convection by Sullivan and Ahlers [Phys. Rev. A **38**, 3143 (1988)]. [S0031-9007(98)06353-4]

PACS numbers: 47.20.Ky, 05.45.+b, 47.20.–k

Bursts of activity, be they regular or irregular, are a common occurrence in physical and biological systems. For example, they are characteristic of neural activity where they are typically associated with bistability [1]. In hydrodynamics the term is used to describe bursts of turbulent activity interspersed with laminar behavior [2] or the intermittent destruction of coherent structures in the wall region of turbulent boundary layers [3]. A different type of burstlike behavior, not associated with turbulence, has been reported in binary fluid convection, for example, in ^{3}He/^{4}He mixtures [4]. Here, despite constant heat input, the convective heat transport (as measured by $N - 1$, where N is the Nusselt number) takes place in a sequence of irregular bursts as shown in Fig. 1. Although different in detail, the bursts in these hydrodynamical systems are all found near the onset ($\epsilon = 0$) of an instability. Moreover, the systems exhibiting this type of behavior all have a substantial degree of symmetry. This suggests that the bursts are not the result of a sequence of bifurcations resulting in more and more complex dynamics but are instead the consequence of a single bifurcation in a system with symmetry. For example, the boundary layer bursts have been successfully attributed to the presence of structurally stable heteroclinic cycles in the interaction between two steady spanwise modes with wave numbers in a 1:2 ratio [3]. With periodic boundary conditions and reflection symmetry in the spanwise direction the system has O(2) symmetry; in the resulting dynamical systems description the fixed points are associated with the presence of coherent structures and the recurrent excursions along the heteroclinic cycle with the observed bursts. In this Letter we identify a different mechanism for producing bursts. This mechanism provides a natural explanation for the behavior observed in Fig. 1 and also involves the interaction of two modes, but now the modes are oscillatory and the symmetry group is only a reflection Z_2. The resulting bursts have a large dynamical range and are not the result of a structurally stable heteroclinic connection. Moreover, it is the bursts that are associated with the fixed points of a suitable dynamical system, while the excursions between them correspond to the laminar phase.

We consider a slender system with left-right reflection symmetry (such as a narrow rectangular convection cell) undergoing an oscillatory instability from the trivial state. In such a system the first two unstable modes typically have opposite parity under reflection; moreover, because the neutral stability curve for the unbounded system has a parabolic minimum these set in in close succession as the bifurcation parameter is increased. We write the perturbation from the trivial state as

$$\Psi(x, y, t) = \epsilon^{\frac{1}{2}} \mathrm{Re}\{z_+ f_+(x, y) + z_- f_-(x, y)\} + O(\epsilon), \tag{1}$$

where $\epsilon \ll 1$, $f_\pm(-x, y) = \pm f_\pm(x, y)$, and y denotes the transverse variables. The complex amplitudes $z_\pm(t)$ satisfy the equations [5]

$$\dot{z}_\pm = [\lambda \pm \Delta\lambda + i(\omega \pm \Delta\omega)]z_\pm + A(|z_+|^2 + |z_-|^2)z_\pm + B|z_\pm|^2 z_\pm + C\bar{z}_\pm z_\mp^2 . \tag{2}$$

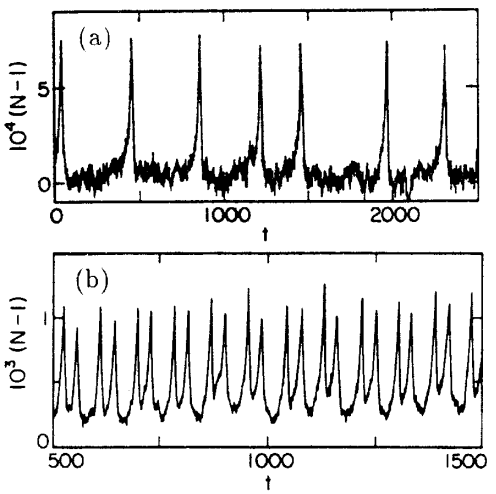

FIG. 1. Bursts in binary fluid convection with separation ratio $S = -0.021$. (a) $\epsilon = 3 \times 10^{-4}$; (b) $\epsilon = 3.6 \times 10^{-3}$. The amplitude of the bursts decreases with increasing ϵ while their frequency increases. After Ref. [4].

 © 1998 The American Physical Society

In these equations the nonlinear terms have identical (complex) coefficients because of an approximate interchange symmetry between the odd and even modes. The resulting D_4 symmetry [5] is weakly broken whenever $\Delta\lambda \neq 0$ and/or $\Delta\omega \neq 0$, a consequence of the finite aspect ratio of the system. To identify the bursts we introduce the change of variables

$$z_{\pm} = \rho^{-\frac{1}{2}} \sin\left(\frac{\theta}{2} + \frac{\pi}{4} \pm \frac{\pi}{4}\right) e^{i(\pm\phi+\psi)/2}$$

and a new timelike variable τ defined by $d\tau/dt = \rho^{-1}$. In terms of these variables Eqs. (2) become

$$\frac{d\rho}{d\tau} = -\rho[2A_R + B_R(1 + \cos^2\theta) + C_R \sin^2\theta \cos 2\phi] - 2(\lambda + \Delta\lambda \cos\theta)\rho^2, \tag{3}$$

$$\frac{d\theta}{d\tau} = \sin\theta[\cos\theta(-B_R + C_R \cos 2\phi) - C_I \sin 2\phi] - 2\Delta\lambda\rho \sin\theta, \tag{4}$$

$$\frac{d\phi}{d\tau} = \cos\theta(B_I - C_I \cos 2\phi) - C_R \sin 2\phi + 2\Delta\omega\rho, \tag{5}$$

where $A = A_R + iA_I$, etc., together with a decoupled equation for $\psi(t)$. The amplitude of the disturbance is measured by $r \equiv |z_+|^2 + |z_-|^2 = \rho^{-1}$; thus $\rho = 0$ corresponds to infinite amplitude states. Equations (3)–(5) show that the restriction to the invariant subspace $\Sigma \equiv \{\rho = 0\}$ is equivalent to taking $\Delta\lambda = \Delta\omega = 0$ in (4) and (5). The resulting D_4-symmetric problem has three generic types of fixed points [6]: u solutions with $\cos\theta = 0, \cos 2\phi = 1$; v solutions with $\cos\theta = 0, \cos 2\phi = -1$; and w solutions with $\sin\theta = 0$. These states correspond to (infinite amplitude) periodic oscillations in time because of the decoupled phase $\psi(t)$. In the binary fluid context the u, v, and w solutions represent mixed parity traveling wave states localized near one of the container walls, mixed parity chevron (or counterpropagating) states, and pure even ($\theta = 0$) or odd ($\theta = \pi$) parity chevron states, respectively [5]. Depending on A, B, and C the subspace Σ may contain additional fixed points and/or limit cycles [6]. In our scenario, a burst occurs for $\lambda > 0$ when a trajectory follows the stable manifold of a fixed point (or a limit cycle) $P_1 \in \Sigma$ that is *unstable* within Σ. The instability within Σ then kicks the trajectory towards another fixed point (or limit cycle) $P_2 \in \Sigma$. If this point has an unstable ρ eigenvalue the trajectory escapes from Σ towards a $\rho > 0$ fixed point (or limit cycle), forming a burst. If $\Delta\lambda$ and/or $\Delta\omega \neq 0$ this fixed point may itself be unstable to perturbations of type P_1 and the process then repeats. The scenario thus requires that at least one of the branches in the D_4-symmetric system be subcritical (P_1) and one supercritical (P_2).

Experimentally, bursts are observed for separation ratios $S = -0.021$ and $S = -0.044$ for ^{3}He/^{4}He mixtures [4] and, for example, $S = -0.032$ for ethanol/water mixtures [7]; these fall within the range $-0.0799 < S < -0.0138$ for typical ^{3}He/^{4}He mixtures and $-0.150 < S < -0.0042$ for typical ethanol/water mixtures for which traveling waves are subcritical and standing waves supercritical [8]. We focus therefore on parameter values for which the u solutions are subcritical and the v, w solutions supercritical when $\Delta\lambda = \Delta\omega = 0$. When $\Delta\lambda$ and/or $\Delta\omega \neq 0$ two types of oscillations in (θ, ϕ) are possible: rotations and librations (see Fig. 2). These oscillations are coupled to excursions in amplitude. Figure 3 shows the resulting sequence of large amplitude bursts; these arise from repeated excursions towards the infinite amplitude ($\rho = 0$) u solutions. Irregular bursts are also readily generated: Figure 4 shows bursts arising from chaotic rotations. Figures 5(a) and 5(b) provide a partial summary of the different solutions of Eqs. (3)–(5) and their stability properties; a detailed description of the origin of the complexity revealed in these figures is given elsewhere [9]. Here we focus on its physical consequences.

In Fig. 6 we show the (x, t) plots of the sequences of bursts corresponding to the trajectories shown in Fig. 2. The bursts in Fig. 6(a) are generated as a result of successive visits to *different* but symmetry-related infinite amplitude u solutions [cf. Fig. 2(a)]; in Fig. 6(b) the generating trajectory makes repeated visits to the *same* infinite amplitude u solution [cf. Fig. 2(b)].

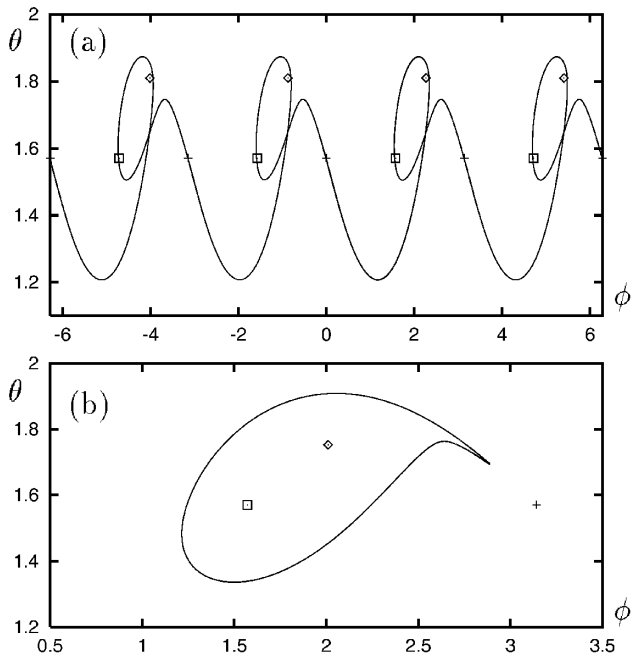

FIG. 2. Stable periodic rotations and librations at (a) $\lambda = 0.1$ and (b) $\lambda = 0.1253$, respectively, for $\Delta\lambda = 0.03$, $\Delta\omega = 0.02$, $A = 1 - 1.5i$, $B = -2.8 + 5i$, $C = 1 + i$. The + signs indicate infinite amplitude u solutions responsible for the bursts, while squares and diamonds indicate infinite amplitude v solutions and finite amplitude periodic oscillations.

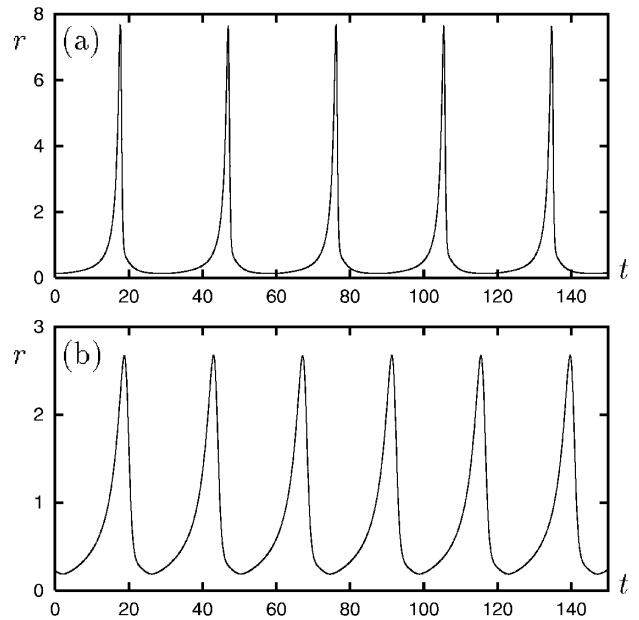

FIG. 3. Time series showing periodic bursts corresponding to the trajectories in Fig. 2.

These plots are constructed using the approximate expression $f_{\pm} = \{e^{-\gamma x + ix} \pm e^{\gamma x - ix}\} \cos \frac{\pi x}{L}$, where $\gamma = 0.15 + 0.025i$, $L = 80$, and $-\frac{L}{2} \leq x \leq \frac{L}{2}$. The former state is typical of the blinking state identified in binary fluid convection in rectangular containers [10]; it is likely that the irregular bursts shown in Fig. 1 are due to such a state. The latter is a new state which we call a *winking* state; winking states may be stable but often coexist with stable chevronlike states which are more likely to be observed in experiments in which the Rayleigh number is ramped upwards (see Fig. 5). The blinking state in Fig. 6(a) closely resembles the state reported in Ref. [7] and attributed there to burstlike solutions of a single complex Ginzburg-Landau equation with a destabilizing nonlinearity and periodic boundary conditions. However, the data in Ref. [7] indicate that when $L = 40.6$ the bursts

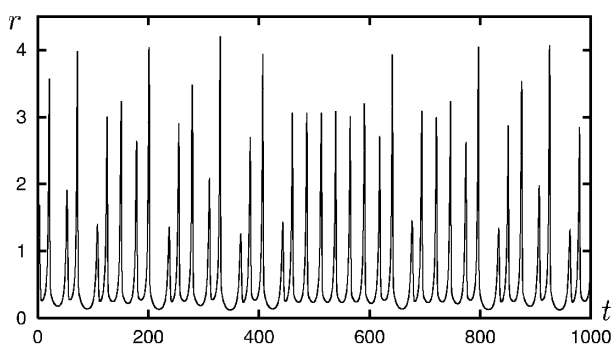

FIG. 4. Time series showing bursts from chaotic rotations at $\lambda = 0.072$. This solution describes a chaotic blinking state because the trajectory makes successive visits to different but symmetry-related infinite amplitude u solutions.

are localized preferentially near the sidewalls and hence that the sidewalls play a critical role at this L provided $\epsilon \lesssim 0.009$. For larger ϵ (for the given L) or larger L (for the given ϵ) the burst mechanism described here is likely superseded by that outlined in Ref. [7].

The bursts described here are the result of oscillations in amplitude between two modes of opposite parity and "frozen" spatial structure. They occur very close to onset ($\epsilon = 3 \times 10^{-4}$ in Ref. [4]) so that the spatial structure is well approximated by the linear eigenfunction. The presence of bursts requires at least one of the branches in the D_4-symmetric system to be subcritical. Moreover, a large aspect ratio L is required for the approximate interchange symmetry to hold; if the size of the D_4 symmetry-breaking terms $\Delta\lambda$, $\Delta\omega$ is increased too much the bursts fade away and are replaced by smaller amplitude, higher frequency states. For example, if $\Delta\omega \gg \Delta\lambda$ averaging eliminates the C terms responsible for the bursts. Thus bursts will not be present if L is

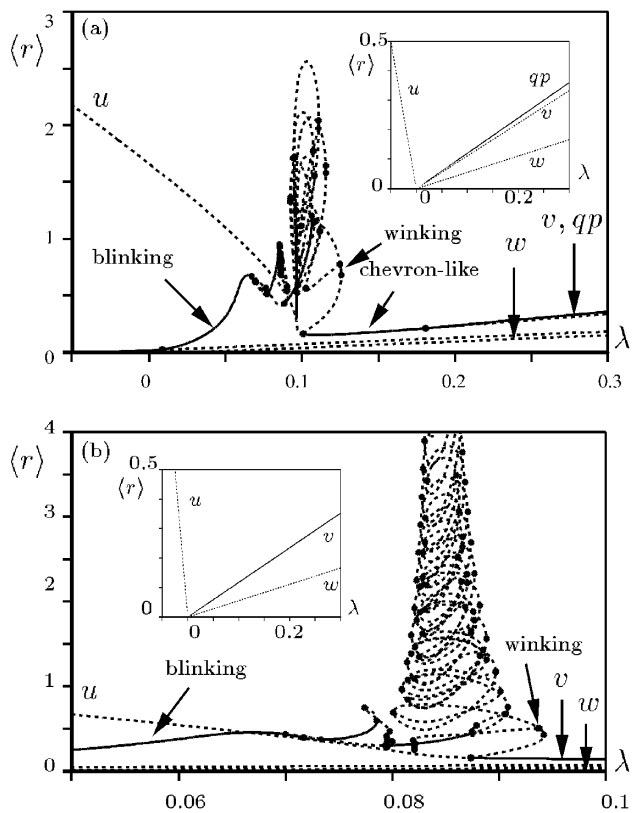

FIG. 5. Partial bifurcation diagrams for (a) $C = 1 + i$ and (b) $C = 0.9 + i$ with the remaining parameters as in Fig. 2 showing the time average of r for different solutions as a function of λ. Solid (dashed) lines indicate stable (unstable) solutions. The branches labeled u, v, w, and qp (quasiperiodic) may be identified in the limit of large $|\lambda|$ with branches in the corresponding diagrams when $\Delta\lambda = \Delta\omega = 0$ (insets). All other branches correspond to bursting solutions which may be blinking or winking states. Circles, squares, and diamonds in the diagram indicate Hopf, period-doubling, and saddle-node bifurcations, respectively.

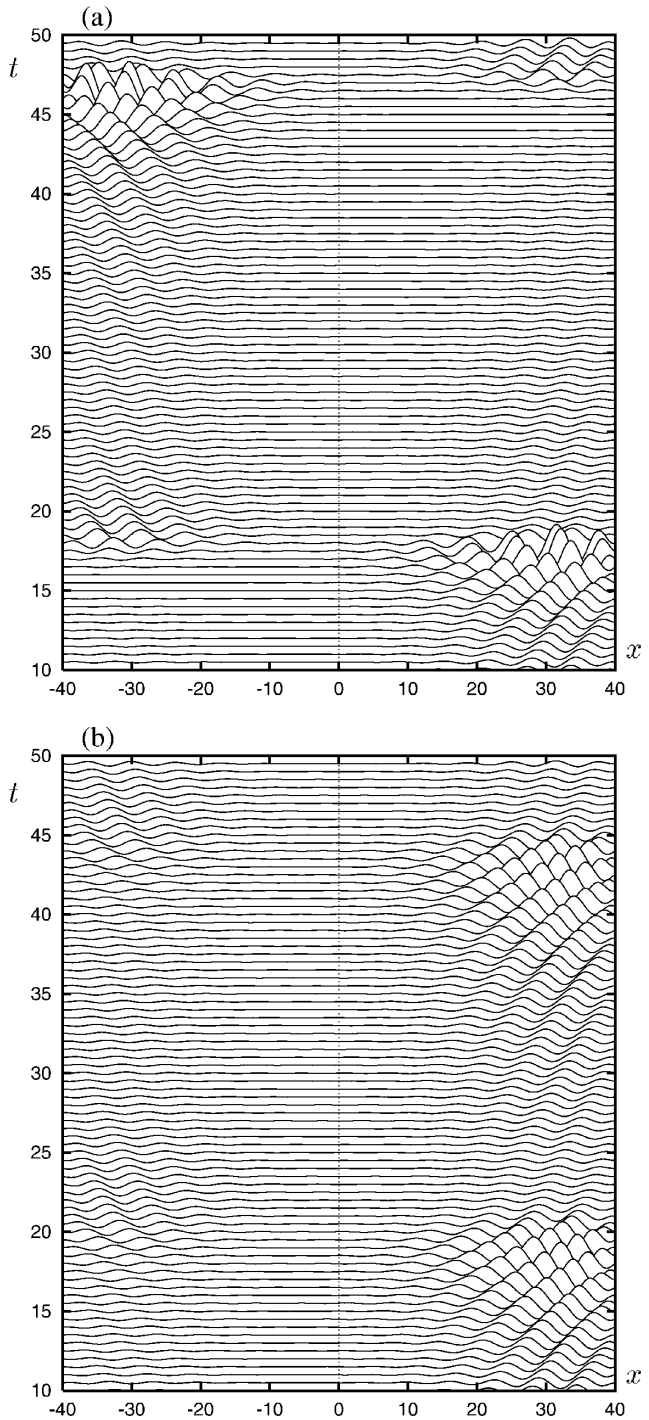

FIG. 6. The perturbation Ψ from the trivial state represented in a space-time plot showing (a) a periodic blinking state (in which successive bursts occur at opposite sides of the container) from the trajectory in Fig. 2(a), and (b) the periodic winking state (in which successive bursts occur at the same side of the container) for the trajectory in Fig. 2(b).

too small or ϵ too large; cf. Fig. 1. The mechanism described here is similar in spirit to that put forward by Newell *et al.* [11] but differs in that ours applies in fully dissipative driven systems and relies on the presence of reflection symmetry. However, both involve global connections to infinity and hence are capable of describing bursts of arbitrarily large dynamical range in contrast to the mechanism of Ref. [3]. Consequently it is possible that the burst amplitude can become large enough that secondary instabilities not captured by the ansatz (1) can be triggered. Such instabilities could occur on very different scales and result in *turbulent* rather than just large amplitude bursts. It should be emphasized that the physical amplitude of the bursts is $O(\epsilon^{\frac{1}{2}})$ and so approaches zero as $\epsilon \downarrow 0$; cf. Eq. (1). Thus despite their large dynamical range the bursts are fully and correctly described by the asymptotic expansion that leads to Eqs. (2). In particular, the mechanism is robust with respect to the addition of small fifth order terms [12]. We expect that the mechanism identified here will be detected in other systems with approximate D_4 symmetry such as lasers [13], spring-supported fluid-conveying tubes [14], and the Faraday system [15], as it has been in dynamo theories of magnetic field generation in the Sun [16].

This work was supported by the National Science Foundation under Grant No. DMS-9703684.

[1] J. D. Murray, *Mathematical Biology* (Springer-Verlag, New York, 1989).
[2] K. Coughlin and P. S. Marcus, Phys. Rev. Lett. **77**, 2214 (1996).
[3] P. Holmes, J. L. Lumley, and G. Berkooz, *Turbulence, Coherent Structures, Dynamical Systems and Symmetry* (Cambridge University Press, Cambridge, England, 1996).
[4] T. S. Sullivan and G. Ahlers, Phys. Rev. A **38**, 3143 (1988).
[5] A. S. Landsberg and E. Knobloch, Phys. Rev. E **53**, 3579 (1996).
[6] J. W. Swift, Nonlinearity **1**, 333 (1988).
[7] E. Kaplan, E. Kuznetsov, and V. Steinberg, Phys. Rev. E **50**, 3712 (1994).
[8] T. Clune and E. Knobloch, Physica (Amsterdam) **61D**, 106 (1992).
[9] E. Knobloch and J. Moehlis (to be published).
[10] P. Kolodner, C. M. Surko, and H. Williams, Physica (Amsterdam) **37D**, 319 (1989); V. Steinberg, J. Fineberg, E. Moses, and I. Rehberg, *ibid.* **37**, 359 (1989). Related behavior occurs in the Taylor-Couette system [C. D. Andereck, S. S. Liu, and H. L. Swinney, J. Fluid Mech. **164**, 155 (1986)] and doubly diffusive convection [A. A. Predtechensky *et al.,* Phys. Fluids **6**, 3923 (1994)].
[11] A. C. Newell, D. A. Rand, and D. Russell, Phys. Lett. A **132**, 112 (1988); Physica (Amsterdam) **33D**, 281 (1988).
[12] J. Moehlis and E. Knobloch (unpublished).
[13] Q. Feng, J. V. Moloney, and A. C. Newell, Phys. Rev. A **50**, R3601 (1994).
[14] A. Steindl and H. Troger, Nonlinear Dyn. **8**, 161 (1995).
[15] F. Simonelli and J. P. Gollub, J. Fluid Mech. **199**, 471 (1989).
[16] E. Knobloch and A. S. Landsberg, Mon. Not. R. Astron. Soc. **278**, 294 (1996); E. Knobloch, S. M. Tobias, and N. O. Weiss, *ibid.* (to be published).