BURSTS

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1. Introduction

A thorough understanding of imperfections is critical for many engineering applications. At the linear level, local imperfections can trap modes and be responsible for a new class of potential instabilities. At the weakly nonlinear level, imperfections are known to play an important role near degeneracies in parameter space. Nominally symmetric systems are always degenerate in this sense since the presence of symmetries typically eliminates certain terms from the amplitude equations describing the evolution of instabilities in such systems. Consequently, such systems are almost always sensitive to small symmetry-breaking imperfections. Particularly dangerous are imperfections that destroy continuous symmetries, such as translation or rotation invariance. Such imperfections are typically responsible for the introduction of global bifurcations into the dynamics, and these are likely to be responsible for the appearance of chaos in the imperfect system (Knobloch 1996). However, in certain cases the loss of discrete symmetry can have a similar effect, at least if the symmetry group is large enough. This is the case in the class of systems discussed below.

We consider here systems with approximate D_4 symmetry undergoing a Hopf bifurcation. This symmetry arises frequently in applications. As an example, consider a system of partial differential equations defined on a square domain, and suppose that the equations (and boundary conditions) are invariant under reflections and rotations of the square by 90°. A Hopf bifurcation in such a system will be described by the amplitude equations for the Hopf bifurcation with D_4 symmetry. In normal form these equations have an additional S^1 phase shift symmetry which allows an essentially complete discussion of their dynamics. In particular, it is possible to show that the third order truncation of this normal form cannot exhibit chaotic dynamics. The proof proceeds by showing that the resulting equations can be written as a dynamical system defined on the surface of a three-dimensional sphere in phase space, together with two other decoupled equations (Swift 1988). When the D_4 symmetry is broken by turning the domain into a rectangular one, albeit with a nearly square cross-section, this decoupling no longer occurs, and the dynamics becomes fundamentally three-dimensional. A closely related problem is provided by the Faraday system in a nearly square container. In this system, gravity-capillary waves are excited on the surface of a viscous fluid by vertical vibration of the container, usually as a result of a subharmonic resonance. Because of the parametric forcing the S^1 symmetry is now absent, but the amplitude equations describing the interaction of roll-like states oriented parallel to the sides continue to have approximate D_4 symmetry. In a square container, careful experiments by Simonelli and Gollub (1989) uncovered no chaotic dynamics in this system. On the other hand, in a rectangular but nearly square container the situation is quite different. Here Simonelli and Gollub uncovered the presence of a new class of oscillations, hereafter called "bursts". These are oscillations in the *amplitudes* of the two competing modes in which the energy builds up to a high value before undergoing an abrupt collapse to a small amplitude state. Such bursts can occur either periodically or irregularly, depending on parameters. It is this behavior that is of interest in the present paper. Similar behavior was also noted in experiments on convection in He^3/He^4 mixtures at cryogenic temperatures (Sullivan and Ahlers 1988). In these experiments, performed in a $34:6.9:1$ cell, the heat transport through the system exhibited irregular large amplitude bursting only 0.03% above threshold for the convective instability. Here, however, the origin of the approximate D_4 symmetry is quite different. Because of the slender shape of the container we may idealize the system as effectively two-dimensional. Such a system will undergo an oscillatory instability to either a mode of even or odd parity with respect to reflection in the extended direction. In the (formal) limit in which the aspect ratio of the system is allowed to become large, the distinction between even and odd modes is lost, and the amplitude equations describing the interaction of these two modes acquire an additional "interchange" symmetry (Landsberg and Knobloch 1996). As discussed further below, this interchange symmetry together with the reflection generates the group D_4. In a finite container this symmetry is never exact, however, since one or other of the two competing modes sets in first. Consequently the mode interaction in a system of large but finite extent is described by the normal form equations for the Hopf bifurcation with broken D_4 symmetry. This picture is supported by numerical simulations of the partial differential equations describing two-dimensional binary fluid convection in systems with aspect ratio $L = 16.25$ (Jacqmin and Heminger 1994). These simulations also reveal the presence of bursts, typically irregular and sometimes of very large amplitude, but in addition provide diagnostics that demonstrate convincingly that they involve primarily two adjacent modes of opposite parity. These results provide the primary motivation for studying the Hopf bifurcation with broken D_4 symmetry.

2. Derivation of the Equations

In this section we sketch the derivation of the amplitude equations describing the interaction of adjacent Hopf modes in a one-dimensional container of length L with identical boundary conditions at $x = \pm L/2$. Such a system has a reflection symmetry about $x = 0$; the primary Hopf modes are either even or odd under under this reflection (Dangelmayr and Knobloch 1991). We consider the interaction of an even mode with an adjacent odd mode in the formal limit $L \to \infty$. Let (z_+, z_-) be the complex amplitudes of the two modes. The requirement that a reflected state also be a state of the system translates into the requirement that the amplitude equations be equivariant with respect to the group action

$$\kappa_1 : (z_+, z_-) \to (z_+, -z_-). \tag{1}$$

Moreover, as argued by Landsberg and Knobloch (1996), the equations for the formally infinite system cannot distinguish between the two modes, i.e. in this limit the amplitude equations must also be equivariant with respect to the group action

$$\kappa_2 : (z_+, z_-) \to (z_-, z_+). \tag{2}$$

These two operations generate the group $D_4 = \langle \kappa_1, \kappa_2 \rangle$. For a container with large but finite length, this symmetry will be weakly broken; in particular, the even and odd modes may become unstable at slightly different Rayleigh numbers and with slightly different frequencies. The resulting equations are thus close to those for a 1:1 resonance, but with a special structure dictated by the proximity to D_4 symmetry. Because of the normal form symmetry

$$\hat{\sigma} : (z_+, z_-) \to \mathrm{e}^{i\sigma}(z_+, z_-), \qquad \sigma \in [0, 2\pi), \tag{3}$$

the resulting equations have an additional S^1 symmetry. If these equations are truncated at third order we obtain (Landsberg and Knobloch 1996)

$$\dot{z}_+ = [\lambda + \Delta\lambda + i(\omega + \Delta\omega)]z_+ + A(|z_+|^2 + |z_-|^2)z_+ + B|z_+|^2 z_+ + C\bar{z}_+ z_-^2 \tag{4}$$

$$\dot{z}_- = [\lambda - \Delta\lambda + i(\omega - \Delta\omega)]z_- + A(|z_+|^2 + |z_-|^2)z_- + B|z_-|^2 z_- + C\bar{z}_- z_+^2 \tag{5}$$

Here $\Delta\omega$ measures the difference in frequency between the two modes at onset, and $\Delta\lambda$ measures the difference in their linear growth rates. Under appropriate nondegeneracy conditions (which we assume here) we may neglect all interchange symmetry-breaking contributions to the nonlinear terms. In terms of new variables defined by

$$z_+ = \rho^{-1/2}\cos(\theta/2)e^{i(\phi+\psi)/2}, \qquad z_- = \rho^{-1/2}\sin(\theta/2)e^{i(-\phi+\psi)/2}, \tag{6}$$

and a new time τ defined by $d\tau/dt = \rho^{-1}$, these equations take the more convenient form

$$\frac{d\rho}{d\tau} = -\rho[2A_R + B_R(1+\cos^2\theta) + C_R\sin^2\theta\cos 2\phi] - 2(\lambda + \Delta\lambda\cos\theta)\rho^2 \tag{7}$$

$$\frac{d\theta}{d\tau} = \sin\theta[\cos\theta(-B_R + C_R\cos 2\phi) - C_I\sin 2\phi] - 2\Delta\lambda\sin\theta\rho \tag{8}$$

$$\frac{d\phi}{d\tau} = \cos\theta(B_I - C_I\cos 2\phi) - C_R\sin 2\phi + 2\Delta\omega\rho, \tag{9}$$

where $A = A_R + iA_I$, etc. Equations (7-9) are invariant under the operations $\theta \to \theta + 2\pi$ and $\phi \to \phi + \pi$, symmetries which are related to the (broken) $D_4 \times S^1$ symmetry of the full system. A consequence of these symmetries is that if $(\rho_0, \theta_0, \phi_0)$ is a fixed point of equations (7-9), then so are the points $(\rho_0, \theta_0 + m\pi, \phi_0 + 2n\pi)$, where m and n are integers. This is so also for periodic solutions. We say that such fixed points and periodic solutions are *symmetry-related.*

The behavior of the decoupled variable ψ may be found by solving the appropriate differential equation for ψ evaluated at the solutions to the three-dimensional system. In particular, ψ (modulo 4π) is periodic for fixed points and periodic solutions to the three-dimensional system; thus, fixed points and periodic solutions in the three-dimensional system correspond to periodic solutions and tori in the full four-dimensional system (4,5), respectively. In the following we use the variable $r \equiv \rho^{-1} = |z_+|^2 + |z_-|^2$ as a useful measure of the energy in a burst.

3. Fixed points and periodic solutions of the three-dimensional system

3.1. THE PERFECT SYSTEM $\Delta\lambda = \Delta\omega = 0$

The D_4-symmetric equations $\Delta\lambda = \Delta\omega = 0$ have been analyzed by Swift (1988). In this case the $(\theta(\tau), \phi(\tau))$ equations decouple from the rest, and describe dynamics on the surface of a sphere of variable radius. For fixed points (θ_0, ϕ_0) of this two-dimensional system this radius becomes constant as $\tau \to \infty$, and depends on the quantity

$$F(\theta, \phi) = 2A_R + B_R(1+\cos^2\theta) + C_R\sin^2\theta\cos 2\phi. \tag{10}$$

Specifically,

(a) if $\lambda < 0, F(\theta_0, \phi_0) < 0$ then $r \to 0$,
(b) if $\lambda F(\theta_0, \phi_0) < 0$ then $r \to r_0 > 0$,
(c) if $\lambda > 0, F(\theta_0, \phi_0) > 0$ then $r \to \infty$.

Provided case (b) holds, there are three types of fixed points (θ_0, ϕ_0) with nontrivial symmetry which exist for all coefficient values (Swift, 1988). In addition there are open regions of coefficient space with nonsymmetric fixed points (that is, fixed points with trivial isotropy) and others with a unique limit cycle $(\theta^*(\tau), \phi^*(\tau))$. For the limit cycles we define

$$\bar{F} = \frac{1}{T_\tau}\int_0^{T_\tau} F(\theta^*(\tau'), \phi^*(\tau'))d\tau', \tag{11}$$

where T_τ is the period, and conclude (cf. van Gils and Silber 1995) that
(a) if $\lambda < 0, \bar{F} < 0$ then $r \to 0$,
(b) if $\lambda \bar{F} < 0$, there exists a nonzero, finite $r(\tau)$ with $r(\tau + T_\tau) = r(\tau)$,
(c) if $\lambda > 0, \bar{F} > 0$ then $r \to \infty$.

Because the associated spherical system (8,9) is two-dimensional, no complex dynamics is possible unless the S^1 normal form symmetry is broken, as in the Faraday system. We do not pursue this possibility here, and instead focus on the effects of breaking the D_4 symmetry. We find that the possibilities (c) are responsible for the bursts present in this system.

3.2. THE IMPERFECT SYSTEM $\Delta\lambda \neq 0$ AND/OR $\Delta\omega \neq 0$

For the equations with broken D_4 symmetry ($\Delta\lambda \neq 0$ and/or $\Delta\omega \neq 0$) only the fixed points with even and odd parity remain as primary branches; the analogs of the remaining primary branches may bifurcate in secondary bifurcations from these, and are most easily found numerically (Landsberg and Knobloch 1996). However, there is another class of fixed points as well; these are crucial for understanding the bursting behavior. Their existence follows from the restriction of equations (7-9) to the invariant subspace $\rho = 0$. The resulting equations are identical to equations (8,9) with $\Delta\lambda = \Delta\omega = 0$; thus the fixed points of the associated spherical system that governs the dynamics of the perfect system continue to have significance for the imperfect system but now correspond to *infinite* amplitude fixed points. Infinite amplitude nonsymmetric fixed points and infinite amplitude limit cycles may also exist depending on the values of the parameters, exactly as in the perfect problem. Indeed, for $\Delta\lambda = \Delta\omega = 0$, infinite amplitude fixed points are amplitude-stable for $\lambda > 0, F(\theta_0, \phi_0) > 0$, while infinite amplitude limit cycles are amplitude-stable for $\lambda > 0, \bar{F} > 0$.

It is a simple matter to obtain the eigenvalues s_1, $s_\pm$ of the fixed points (ρ, θ, ϕ) of equations (7-9) and the corresponding eigenvectors. For the infinite amplitude fixed point $(\rho, \theta, \phi) = (0, \frac{\pi}{2}, 0)$ we obtain

$$s_1 = -2A_R - B_R - C_R, \quad s_\pm = \frac{1}{2}\left\{B_R - 3C_R \pm \sqrt{B_R^2 + 8B_I C_I - 8C_I^2 + 2B_R C_R + C_R^2}\right\},$$

with eigenvectors

$$\left(\frac{-2A_R^2 - 3A_R B_R - B_R^2 + B_I C_I - C_I^2 + A_R C_R + B_R C_R}{(C_I - B_I)\Delta\lambda + 2(A_R + B_R)\Delta\omega}, \frac{(C_R - 2A_R - B_R)\Delta\lambda + 2C_I\Delta\omega}{(C_I - B_I)\Delta\lambda + 2(A_R + B_R)\Delta\omega}, 1\right),$$

$$\left(0, \frac{1}{2(C_I - B_I)}\left\{B_R + C_R \pm \sqrt{B_R^2 + 8B_I C_I - 8C_I^2 + 2B_R C_R + C_R^2}\right\}, 1\right).$$

The corresponding results for $(0, \frac{\pi}{2}, \frac{\pi}{2})$ follow from the parameter symmetry $C \to -C$. Since s_1 can be negative and the corresponding eigenvector has a nonzero component in the ρ direction, a trajectory starting near the stable manifold will approach infinite amplitude. Moreover, since the eigenvectors of the remaining eigenvalues lie in the invariant plane $\rho = 0$ such a trajectory will evolve, after reaching infinite amplitude, in this plane until it encounters a fixed point (limit cycle) possessing an unstable eigenvalue (Floquet multiplier) that ejects it from it. This is our picture of the bursting mechanism, and we now substantiate it with explicit computations.

4. Genesis of a burst

As an illustrative example, consider the following parameter values:

$$A_R = 1.0, \quad B_R = -2.8, \quad B_I = 5.0, \quad C_R = 1.0, \quad C_I = 1.0,$$

with λ treated as the bifurcation parameter. We do not need to specify A_I which enters into the equation for ψ only. When $\Delta\lambda = \Delta\omega = 0$, the $(\rho_0, \frac{\pi}{2}, \frac{\pi}{2})$ and $\theta = 0$ fixed points are supercritical, while the $(\rho_0, \frac{\pi}{2}, 0)$ is subcritical; all are unstable. For these parameter values nonsymmetric fixed points do not exist. For $\lambda > 0$ there is, in the associated spherical system, a stable periodic solution surrounding the $(\rho_0, \frac{\pi}{2}, \frac{\pi}{2})$ fixed point.

As symmetry breaking parameters we choose $\Delta\lambda = 0.03$ and $\Delta\omega = 0.02$. The results of a detailed numerical study of these parameter values are summarized in the remarkable bifurcation diagram shown in Fig. 1. This figure shows

$$< r > \equiv \frac{1}{T}\int_0^T r dt = \frac{T_\tau}{\int_0^{T_\tau} \rho d\tau} \tag{12}$$

as a function of the bifurcation parameter λ for finite amplitude fixed points and finite amplitude periodic solutions only. Here T is the appropriate period of the solution in the original time t, and T_τ is the period of the solution to equations (7-9) found in terms of the rescaled time τ. In the

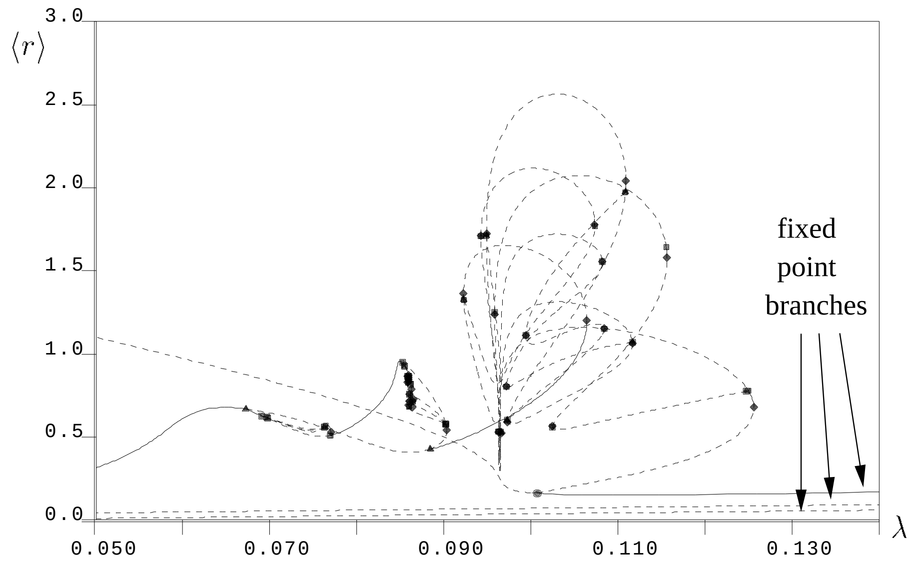

Figure 1. Bifurcation diagram for $A_R = 1.0$, $B_R = -2.8$, $B_I = 5.0$, $C_R = 1.0$, $C_I = 1.0$ and $\Delta\lambda = 0.03$, $\Delta\omega = 0.02$.

figure solid (broken) lines indicate stable (unstable) solutions; circles, diamonds, and squares indicate Hopf, saddle-node, and period-doubling bifurcations, respectively. Many period-doubled branches are omitted. The periodic branches all correspond to periodic bursts since the corresponding trajectories make periodic excursions towards the invariant plane $\rho = 0$, as shown in Figs. 2 and 3. Note that because the bursts are fast events in the original time t the average $\langle r \rangle$ for a sequence of large amplitude bursts may in fact be quite small. At some values of λ, such as $\lambda = 0.072$, no stable periodic branches are present, and irregular bursts are found as shown in Fig. 4. This figure includes a plot of successive maxima against one another; the map appears one-dimensional, and shows unambiguously that the bursts are chaotic. A number of global bifurcations in which a periodic solution approaches a finite amplitude fixed point are found to occur near $\lambda = 0.0965$ and manifest themselves as cusps in Fig. 1.

Fig. 1 omits a very important class of global bifurcations involving *infinite* amplitude solutions, i.e. solutions with $\rho = 0$. For these coefficients, $s_1(0, \frac{\pi}{2}, 0) = -0.2$ with eigenvector $(-3.96, -0.49, 1)$. Moreover, $s_+(0, \frac{\pi}{2}, 0) = 0.068$ and $s_-(0, \frac{\pi}{2}, 0) = -5.87$; consequently the infinite amplitude fixed point is a saddle point whose unstable manifold forms a structurally stable connection with an infinite amplitude limit cycle around $(0, \frac{\pi}{2}, \frac{\pi}{2})$. This limit cycle is attracting in the $\rho = 0$ invariant plane but

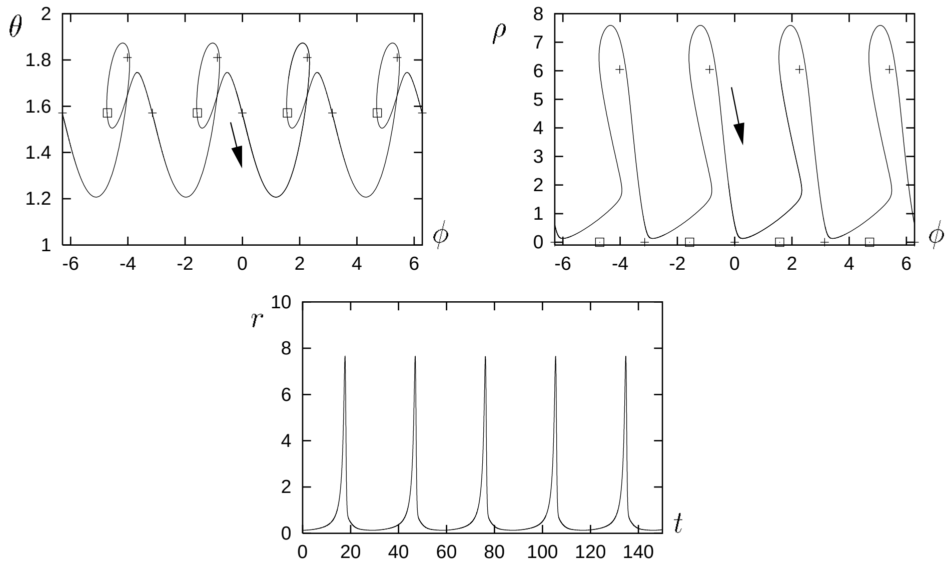

Figure 2. Periodic bursts from successive excursions towards symmetry-related infinite amplitude fixed points. For this burst sequence $\lambda = 0.1$, $\langle r \rangle \approx 0.7$.

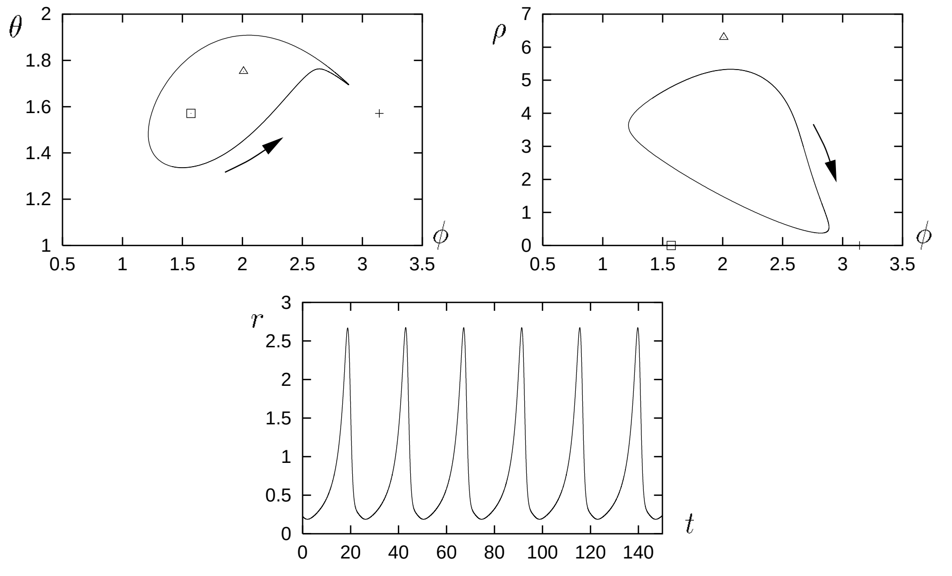

Figure 3. Periodic bursts from successive excusrions towards the same infinite amplitude fixed point. For this burst sequence $\lambda = 0.1253$, $\langle r \rangle \approx 0.75$.

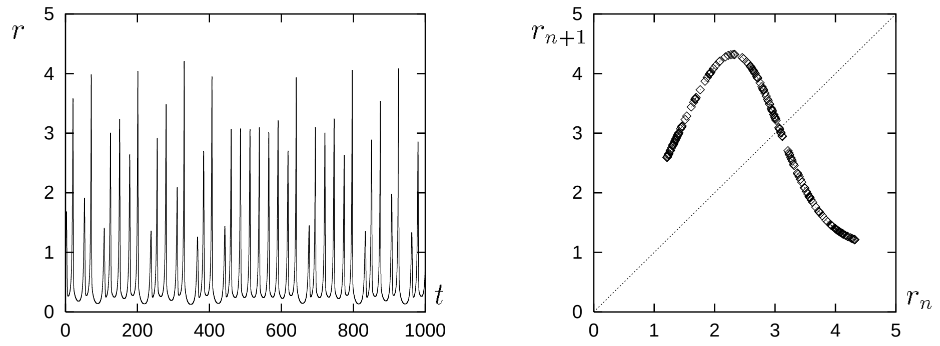

Figure 4. Chaotic bursts for $\lambda = 0.072$.

for these coefficient values possesses an unstable Floquet multiplier in the ρ direction and hence has a two-dimensional unstable manifold. These manifolds together describe a single infinite amplitude burst (see Fig. 5). Recurrent bursts occur if this unstable manifolds intersects the stable manifold of the fixed point or its translates (cf. Figs. 2,3). This is a codimension-one phenomenon, and can be studied using a Shil'nikov-like analysis; this analysis bears substantial similarity to that carried out by Hirschberg and Knobloch (1993) for the Shil'nikov-Hopf bifurcation, and is particularly relevant to very large bursts such as those present at $\Delta\lambda = 0.06$, $\Delta\omega = -0.01$ (Fig. 6). A similar geometrical scenario is responsible for the generation of chaotic traveling waves in systems with O(2) symmetry (Knobloch and Moore 1991). In the present case, however, the trajectories typically also approach close to a λ-dependent finite amplitude fixed point and consequently the analysis of the global bifurcations at infinity provides only a partial description of the bifurcation diagram of Fig. 1. Further details can be found in Moehlis and Knobloch (1997).

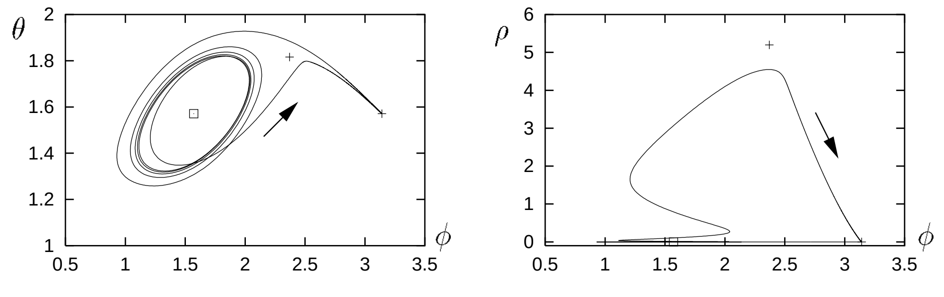

Figure 5. Heteroclinic cycle involving an infinite amplitude fixed point and an infinite amplitude limit cycle for $\lambda = 0.0974$.

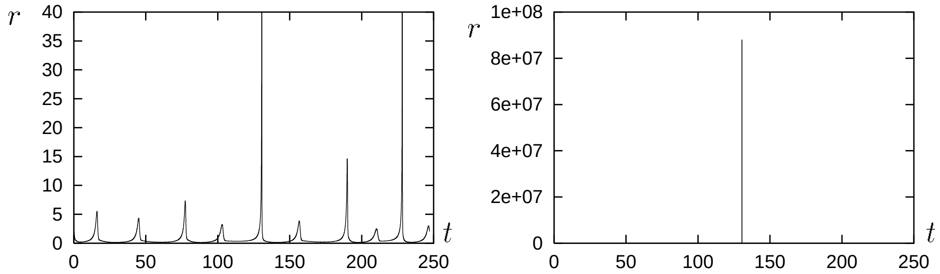

Figure 6. Very large amplitude burst for $\Delta\lambda = 0.06$, $\Delta\omega = -0.01$, $\lambda = 0.1$.

5. Conclusion

The bursting mechanism outlined above arises in a natural way in slender systems supporting oscillations of even and odd parity. The resulting dramatic response can lead to material fatigue or, in the fluid context, to the breakdown of laminar motion into intermittent turbulent bursts. Although neither development is captured by the "frozen" spatial structure responsible for the burst mechanism, we have invoked such potential applications in naming this interesting phenomenon.

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