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region of quasiperiodic and chaotic behavior was present near onset. When these oscillations first appear they take the form of relaxation oscillations in which the surface of the fluid remains flat for a long time before a "large wave grows, reaches a maximum, and decays, all in a time short compared with the period". The duration of the spikes is practically independent of the forcing amplitude, while the interspike period appears to diverge as the forcing amplitude *decreases*. The spikes themselves possess the characteristic asymmetry seen in Figs. 2 and 3. This behavior occurs when the forcing frequency lies *below* the resonance frequency of the square container, i.e., precisely when D_4-symmetric problem has a *subcritical* branch. Irregular bursts are also found, depending on parameters, but these are distinct from the chaotic states found by Nagata [67] far from threshold and present even in a square container. Crawford [68, 69] points out that depending on the mode interaction the dynamics in a square container and a nonsquare container with D_4 symmetry may be substantially different.

5. Discussion. In this article we have seen that there are many different mechanisms responsible for bursting in hydrodynamical systems. Thus no single mechanism can be expected to provide a universal explanation for the observations. Although the mechanisms we have described all rely on the presence of global bifurcations there are important differences among them. For example, the bursts in the wall region of a turbulent boundary layer described in section 2.1 are due to a (structurally stable) heteroclinic cycle connecting fixed points with *finite* amplitude; such a cycle leads to bursts with a limited dynamical range. In contrast in the mechanism of section 3 the dynamical range is unlimited. Moreover, the role of the fixed points is different: in the former the bursts are associated with the *excursions* between the fixed points while in the latter the bursts are associated *with* the fixed points. Because of the structural stability of the cycle the time between successive bursts in the turbulent boundary layer will increase without bound unless the stochastic pressure term is included; such a stochastic term is not required in the mechanism of section 3. In particular in this mechanism the duration of the bursts remains finite despite the fact that they are associated with a heteroclinic connection. This is because of the faster than exponential escape to "infinity" that is typical of this mechanism. This is so also for the mechanism described in section 2.2 although our mechanism applies in fully dissipative driven systems and thus does not rely on the presence of Hamiltonian structure (but it does require the presence of a reflection symmetry). However, both mechanisms involve global connections to infinity and hence are capable of describing bursts of arbitrarily large dynamical range.

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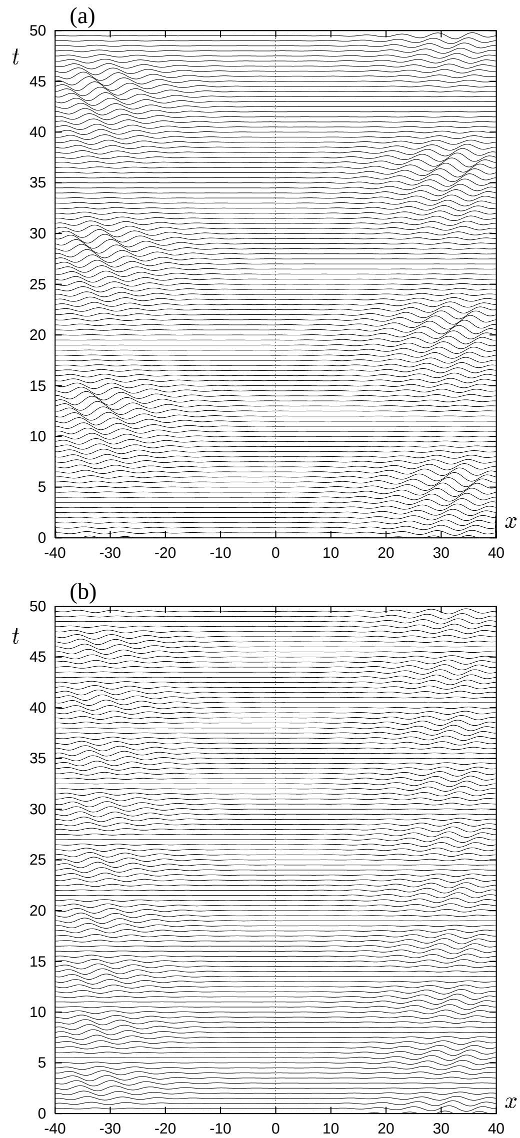

FIG. 7. *The perturbation* Ψ *from the trivial state for parameters chosen as for Fig. 2 except with (a)* $\Delta\omega = 0.1$ *and (b)* $\Delta\omega = 0.5$. *From these and Fig. 6(a) we see that as* $\Delta\omega$ *is increased to large values the bursts fade away and are replaced by smaller amplitude, higher frequency states.*

rectangular container, focusing on the $(3, 2)$, $(2, 3)$ interaction in this system. These modes are degenerate in a square container and only pure and mixed modes were found in this case. In a slightly rectangular container the degeneracy between these modes is broken, however, and in this case a

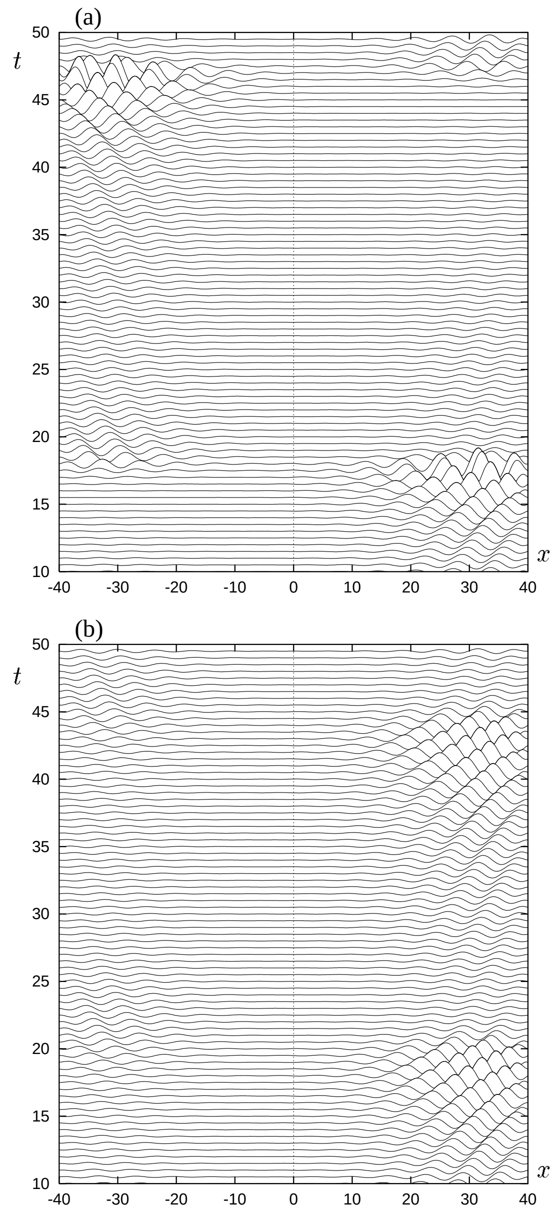

FIG. 6. *The perturbation* Ψ *from the trivial state represented in a space-time plot showing (a) a periodic blinking state (in which successive bursts occur at opposite sides of the container) from the trajectory in Fig. 2, and (b) the periodic winking state (in which successive bursts occur at the same side of the container) for the trajectory in Fig. 3.*

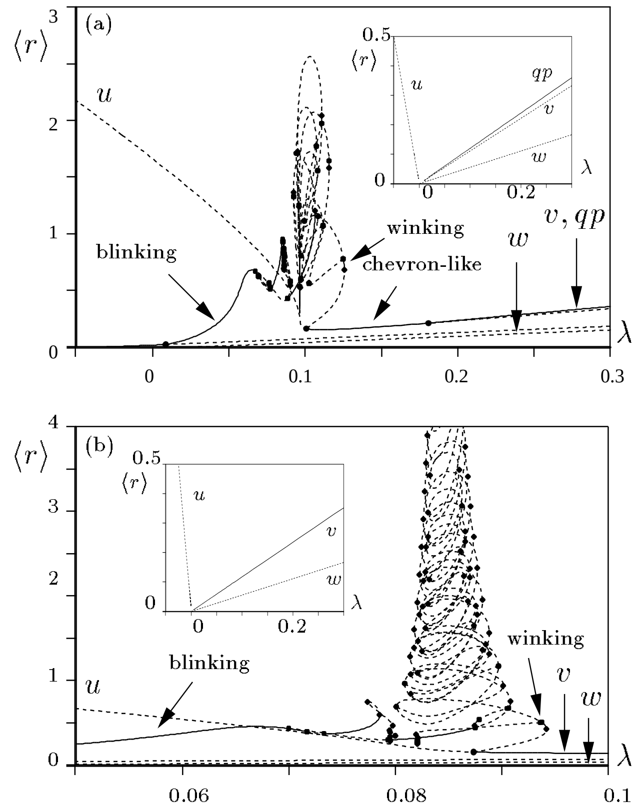

FIG. 5. *Partial bifurcation diagrams for (a) $C = 1 + i$ and (b) $C = 0.9 + i$ with the remaining parameters as in Fig. 2 showing the time-average of r for different solutions as a function of λ. Solid (dashed) lines indicate stable (unstable) solutions. The branches labeled u, v, w, and qp (quasiperiodic) may be identified in the limit of large $|\lambda|$ with branches in the corresponding diagrams when $\Delta\lambda = \Delta\omega = 0$ (insets). All other branches correspond to bursting solutions which may be blinking or winking states. Circles, squares, and diamonds in the diagram indicate Hopf, period-doubling, and saddle-node bifurcations, respectively.*

spatial case) the D_4 symmetry itself would be weakly broken and the mechanism described in the previous section could operate. In this connection it may be interesting that the secondary Hopf bifurcation from spiral vortex flow found in [39] has just such a Floquet multiplier. However, the required reflection symmetry is absent.

Of particular interest is the Faraday system in a nearly square container. In this system gravity-capillary waves are excited on the surface of a viscous fluid by vertical vibration of the container, usually as a result of a subharmonic resonance. Simonelli and Gollub [66] studied the effect of changing the shape of the container from a square to a slightly

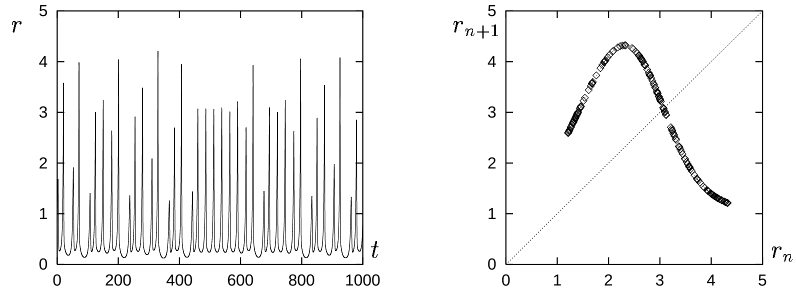

FIG. 4. *Time series and peak-to-peak plot showing bursts from chaotic rotations at $\lambda = 0.072$. This solution describes a chaotic blinking state because the trajectory makes successive visits to different but symmetry-related infinite amplitude u solutions.*

ations, we conclude that bursts will not be present if L is too small or ϵ too large. It is possible that the burst amplitude can become large enough that secondary instabilities not captured by the Ansatz (3.1) can be triggered. Such instabilities could occur on very different scales and result in *turbulent* rather than just large amplitude bursts. It should be emphasized that the physical amplitude of the bursts is $O(\epsilon^{\frac{1}{2}})$ and so approaches zero as $\epsilon \downarrow 0$, cf. eq. (3.1). Thus despite their large dynamical range (i.e., the range of amplitudes during the bursts) the bursts are fully and correctly described by the asymptotic expansion that leads to eqs. (3.2). In particular, the mechanism is robust with respect to the addition of small fifth order terms [53].

4. Other systems with approximate D_4 symmetry. There are a number of other systems of interest where an approximate D_4 symmetry arises in a natural way. These include overstable convection in small aspect ratio containers with nearly square cross-section [59, 60] and more generally any partial differential equation on a nearly square domain describing the evolution of an oscillatory instability, cf. [61]. Other systems in which our bursting mechanism might be detected are lasers [62], spring-supported fluid-conveying tubes [63] and dynamo theories of magnetic field generation in the Sun [64, 65]. More interesting is the possibility that large scale spatial modulation due to distant walls may produce bursting in a fully nonlinear state with D_4 symmetry undergoing a symmetry-breaking Hopf bifurcation. As an example we envisage a steady pattern of fully nonlinear two-dimensional rolls. With periodic boundary conditions with period four times the basic roll period the roll pattern has D_4 symmetry since the pattern is preserved under spatial translations by 1/4 period and a reflection. If such a pattern undergoes a secondary Hopf bifurcation with a *spatial* Floquet multiplier $\exp i\pi/2$ the Hopf bifurcation breaks D_4 symmetry. If the invariance of the basic pattern under translations by 1/4 period is only approximate (this would be the case if the roll amplitude varied on a slow

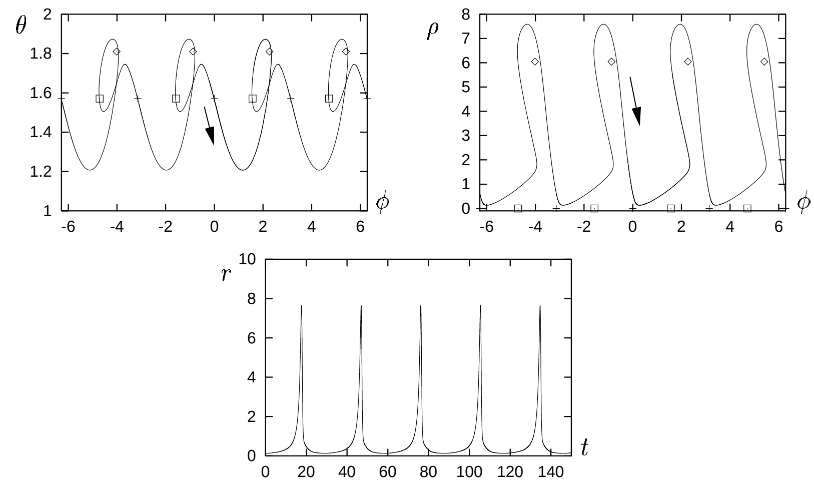

FIG. 2. *Stable periodic rotations at* $\lambda = 0.1$ *for* $\Delta\lambda = 0.03$, $\Delta\omega = 0.02$, $A = 1-1.5i$, $B = -2.8+5i$, $C = 1+i$.

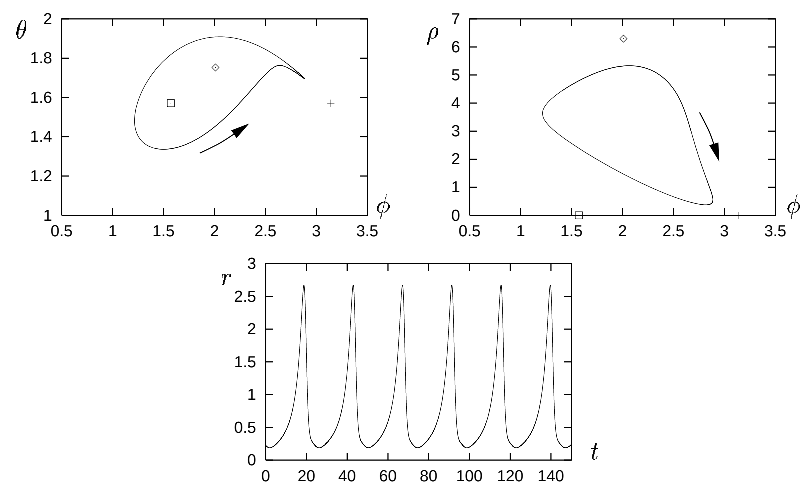

FIG. 3. *As for Fig. 2 but showing stable periodic librations at* $\lambda = 0.1253$.

role as traveling waves in convection [57, 58]. In slender systems, such as the convection system described above or a long Taylor-Couette apparatus, a large aspect ratio L is required for the presence of the approximate D_4 symmetry. If the size of the D_4 symmetry-breaking terms $\Delta\lambda$, $\Delta\omega$ is increased too much the bursts fade away and are replaced by smaller amplitude, higher frequency states (see Fig. 7). Indeed, if $\Delta\omega \gg \Delta\lambda$ averaging eliminates the C terms responsible for the bursts [3]. From these consider-

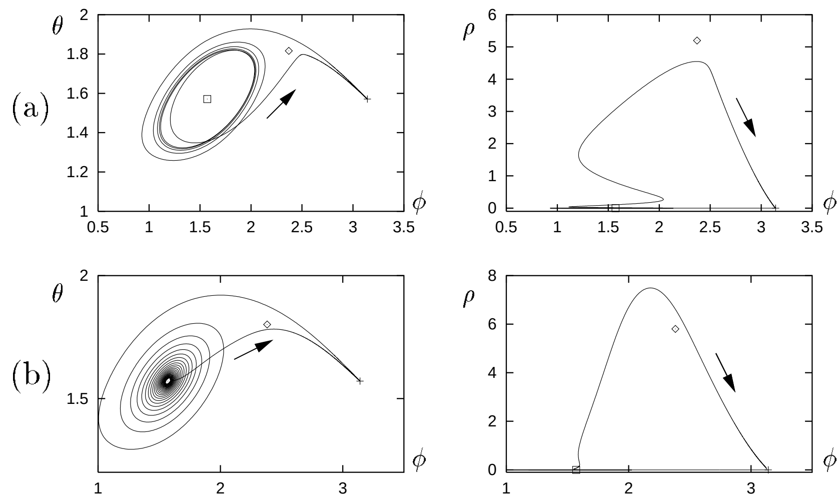

FIG. 1. *Numerically obtained approximate heteroclinic cycles for* $\Delta\lambda = 0.03$*,* $\Delta\omega = 0.02$*,* $A = 1 - 1.5i$*,* $B = -2.8 + 5i$*, and (a)* $C = 1 + i$*, (b)* $C = 0.9 + i$ *present at (a)* $\lambda = 0.0974$ *and (b)* $\lambda = 0.08461$*. The + signs indicate infinite amplitude* u *states responsible for the bursts, while the squares indicate infinite amplitude* v *states and the diamonds finite amplitude states.*

We now focus on the physical manifestation of the bursts. In Fig. 6 we show the solutions of Figs. 2 and 3 in the form of space-time plots using the approximate eigenfunctions

$$f_{\pm}(x) = \left\{e^{-\gamma x + ix} \pm e^{\gamma x - ix}\right\} \cos\frac{\pi x}{L},$$

where $\gamma = 0.15 + 0.025i$, $L = 80$ and $-\frac{L}{2} \le x \le \frac{L}{2}$. The bursts in Fig. 6(a) are generated as a result of successive visits to *different* but symmetry-related infinite amplitude u solutions, cf. Fig. 2; in Fig. 6(b) the generating trajectory makes repeated visits to the *same* infinite amplitude u solution, cf. Fig. 3. The former state is typical of the blinking state identified in binary fluid and doubly diffusive convection in rectangular containers [54]-[56]. It is likely that the irregular bursts reported in [2] are due to such a state. The latter is a new state which we call a *winking* state; winking states may be stable but often coexist with stable chevron-like states which are more likely to be observed in experiments in which the Rayleigh number is ramped upwards (see Fig. 5).

The bursts described above are the result of oscillations in amplitude between two modes of opposite parity and "frozen" spatial structure. Consequently the above burst mechanism applies in systems in which bursts occur very close to threshold. This occurs not only in the convection experiments already mentioned but also in the mathematically identical Taylor-Couette system where counterpropagating spiral vortices play the same

$$-2(\lambda + \Delta\lambda \cos\theta)\rho^2$$

$$\frac{d\theta}{d\tau} = \sin\theta[\cos\theta(-B_R + C_R\cos 2\phi) - C_I \sin 2\phi] - 2\Delta\lambda\ \rho\sin\theta \tag{3.4}$$

$$\frac{d\phi}{d\tau} = \cos\theta(B_I - C_I\cos 2\phi) - C_R\sin 2\phi + 2\Delta\omega\ \rho, \tag{3.5}$$

where $A = A_R + iA_I$, etc. There is also a decoupled equation for $\psi(t)$ so that fixed points and periodic solutions of equations (3.3-3.5) correspond, respectively, to periodic solutions and two-tori in equations (3.2). In the following we measure the amplitude of the disturbance by $r \equiv |z_+|^2 + |z_-|^2 = \rho^{-1}$; thus $\rho = 0$ corresponds to *infinite* amplitude states. Eqs. (3.3-3.5) show that the restriction to the invariant subspace $\Sigma \equiv \{\rho = 0\}$ is equivalent to taking $\Delta\lambda = \Delta\omega = 0$ in (3.4,3.5). The resulting D_4-symmetric problem has three generic types of fixed points [51]: u solutions with $\cos\theta = 0, \cos 2\phi = 1$; v solutions with $\cos\theta = 0, \cos 2\phi = -1$; and w solutions with $\sin\theta = 0$. In the binary fluid context the u, v and w solutions represent mixed parity traveling wave states localized near one of the container walls, mixed parity chevron (or counterpropagating) states, and pure even ($\theta = 0$) or odd ($\theta = \pi$) parity chevron states, respectively [3]. Depending on A, B and C the subspace Σ may contain additional fixed points and/or limit cycles [51]. In our scenario, a burst occurs for $\lambda > 0$ when a trajectory follows the stable manifold of a fixed point (or a limit cycle) $P_1 \in \Sigma$ that is *unstable* within Σ. The instability within Σ then kicks the trajectory towards another fixed point (or limit cycle) $P_2 \in \Sigma$. If this point has an unstable ρ eigenvalue the trajectory escapes from Σ towards a finite amplitude ($\rho > 0$) state, forming a burst. If $\Delta\lambda$ and/or $\Delta\omega \neq 0$ this state may itself be unstable to perturbations of type P_1 and the process then repeats. This bursting behavior is thus associated with a codimension one heteroclinic cycle between the *infinite amplitude* solutions P_1 and P_2 [52, 53]. Examples of such cycles are shown in Fig. 1. Since in such cycles the trajectory reaches infinity in finite time the heteroclinic cycle actually describes bursts of *finite* duration [53].

For the heteroclinic cycle to form it is required that at least one of the branches in the D_4-symmetric system be subcritical (P_1) and one supercritical (P_2). Based on the ^{3}He/^{4}He experiments, we focus on parameter values for which the u solutions are subcritical and the v, w solutions supercritical when $\Delta\lambda = \Delta\omega = 0$ [1]. When $\Delta\lambda$ and/or $\Delta\omega \neq 0$ two types of oscillations in (θ, ϕ) are possible: rotations (see Fig. 2) and librations (see Fig. 3). For $\lambda > 0$ these give rise, under appropriate conditions, to sequences of large amplitude bursts arising from repeated excursions towards the infinite amplitude ($\rho = 0$) u solutions. Irregular bursts are also readily generated: Fig. 4 shows bursts arising from chaotic rotations. Figs. 5(a,b) provide a partial summary of the different solutions of eqs. (3.3-3.5) and their stability properties; much of the complexity revealed in these figures is due to the Shil'nikov-like properties of the heteroclinic cycle [52, 53].

pattern-forming instability, but are present only when the symmetry of the system is weakly broken. Such bursts are typically not associated with turbulence and are therefore easier to describe. Convection in binary fluid mixtures provides a good example. In ^{3}He/^{4}He mixtures in a container with dimensions in the ratio 34 : 6.9 : 1 Sullivan and Ahlers [2] observed that immediately above threshold ($\epsilon \equiv (Ra - Ra_c)/Ra_c = 3 \times 10^{-4}$) convective heat transport may take place in a sequence of irregular bursts of large dynamic range despite constant heat input. Numerical simulations of the two-dimensional equations with no-slip boundary conditions in a container of aspect ratio $L = 16$ suggests that these bursts involve the interaction between the first odd and even modes of the system [45]. An identical description applies to the counterrotating finite length Taylor-Couette system near onset of spiral vortex flow. In both cases we consider a slender system of large (but finite) aspect ratio with left-right reflection symmetry undergoing an oscillatory instability from the trivial state. In such a system the first two unstable modes typically have opposite parity under reflection; moreover, because the neutral stability curve for the unbounded system has a parabolic minimum these typically set in in close succession as the bifurcation parameter is increased. Near threshold the perturbation from the trivial state takes the form

$$\Psi(x,y,t) = \epsilon^{\frac{1}{2}} \mathrm{Re}\, \{z_+ f_+(x,y) + z_- f_-(x,y)\} + O(\epsilon), \tag{3.1}$$

where $\epsilon \ll 1$, $f_\pm(-x,y) = \pm f_\pm(x,y)$, and y denotes transverse variables. The complex amplitudes $z_\pm(t)$ then satisfy the normal form equations [3]

$$\begin{aligned} \dot{z}_\pm =& [\lambda \pm \Delta\lambda + i(\omega \pm \Delta\omega)]z_\pm + A(|z_+|^2 + |z_-|^2)z_\pm \\ &+ B|z_\pm|^2 z_\pm + C\bar{z}_\pm z_\mp^2. \end{aligned} \tag{3.2}$$

In these equations the nonlinear terms have identical (complex) coefficients because of an approximate interchange symmetry between the odd and even modes when $L \gg 1$. The resulting D_4 symmetry (the symmetry group of a square) is weakly broken whenever $\Delta\lambda \neq 0$ and/or $\Delta\omega \neq 0$, a consequence of the finite aspect ratio of the system [3]; in the absence of endwalls $\Delta\lambda = \Delta\omega = 0$ and the D_4 symmetry is exact. Here, as elsewhere [46]-[50], the introduction of small symmetry-breaking terms is responsible for the possibility of complex dynamics in a system that would otherwise behave in a regular manner.

To identify the bursts we introduce the change of variables

$$z_\pm = \rho^{-\frac{1}{2}} \sin\left(\frac{\theta}{2} + \frac{\pi}{4} \pm \frac{\pi}{4}\right) e^{i(\pm\phi+\psi)/2}$$

and a new time-like variable τ defined by $d\tau/dt = \rho^{-1}$. In terms of these variables equations (3.2) become

$$\frac{d\rho}{d\tau} = -\rho[2A_R + B_R(1+\cos^2\theta) + C_R \sin^2\theta \cos 2\phi] \tag{3.3}$$

first state consists of spiral vortices of either odd or even parity with respect to midheight. Slightly above onset the flow resembles interpenetrating spirals (IPS) and these may be intermittently interrupted by bursts of turbulence which fill the entire flow field [38]. In an unbounded system with periodic boundary conditions numerical simulations [39] show that the IPS flow consists of coexisting modes with different axial and azimuthal wavenumbers. This flow is confined primarily to the vicinity of the inner cylinder where the axisymmetric base flow is subject to an inviscid Rayleigh instability. For spatially periodic spiral vortex flow Coughlin and Marcus [39] identify a secondary Hopf bifurcation with the same $m = 4$ as the basic spiral vortex flow but four times the axial wavelength. This bifurcation thus breaks the symmetry of the spiral vortex flow. The secondary instability grows in amplitude and ultimately provides a finite amplitude perturbation to the inviscidly stable flow near the outer cylinder and this triggers a turbulent burst throughout the whole apparatus. During a burst small scales are generated throughout the apparatus leading to a rapid collapse of the turbulence and resumption of the IPS flow; the process can then repeat.

As discussed in Section 3, in a finite Taylor-Couette apparatus there is a natural mechanism for generating bursts. This mechanism does not operate in the axially periodic system, however, and here bursts may be related to the way the secondary instability breaks the symmetry of spiral vortex flow (cf. Section 4).

2.5. Bursts in neural systems. In neural systems, bursting refers to the switching of an observable such as a voltage or chemical concentration between an active state characterized by rapid (spike) oscillations and a rest state. Models of such bursting typically involve singularly perturbed vector fields in which system variables are classified as being "fast" or "slow" depending on whether or not they change significantly over the duration of a single spike. The slow variables may then be thought of as slowly varying parameters for the equations describing the fast variables [40]-[44]. As the slow variables evolve it is possible for the state of the system in the fast variables to change from a stable periodic orbit (corresponding to the active state) to a stable fixed point (corresponding to the rest state) and vice versa; such transitions are often associated with a region of bistability for the periodic orbit and the fixed point but need not be. Mechanisms by which such transitions can occur repeatedly have been classified [40]-[43]. Behavior of the time interval between successive spikes near a transition from the active to the rest state is discussed in [44]; in this paper the presence of a subcritical Hopf-homoclinic bifurcation is also identified as a mechanism for the transition from the active to the rest state.

3. A new mechanism for bursting. In many systems bursting arises as a result of the interaction between spontaneous and forced symmetry breaking. The resulting bursts occur very close to onset of the ing flow is characterized by intermittent bursting [23]-[30]. A burst occurs when the system evolves from a coherent vortex-like modulated traveling wave (MTW) to a spatially disordered state following transfer of energy from large to small scales. The system then relaxes to the vicinity of another symmetry-related MTW state, and the process continues with bursts occurring irregularly but with a well-defined mean period.

The details of what actually happens appear to depend on the value of k because the symmetry of the equation describing the evolution of the Kolmogorov flow depends on k. With 2π-periodic boundary conditions in each direction this symmetry is $\mathrm{D}_{2k}\dot{+}\mathrm{SO}(2)$. In the simplest case, $k=1$, this symmetry group is isomorphic to $\mathrm{O}(2)\times\mathrm{Z}_2$. However, for $k=1$ we must restrict attention to perturbations in x with period larger than 2π in order that the Kolmogorov flow be unstable [31]-[33] and such perturbations are not allowed with 2π-periodic boundary conditions. Alternatively, we may consider the domain $\{-\pi < x \leq \pi, -\pi/k < y \leq \pi/k\}$ with $k > 1$ for which the symmetry group is $\mathrm{O}(2)\times\mathrm{Z}_2$ and perturbations may grow. The unstable modes are then either even or odd under the reflection $(x,y) \rightarrow (-x,-y)$ with respect to a suitable origin. Mode interaction between these two modes can result in the following sequence of transitions [15]: the Kolmogorov flow loses stability to an even mode, followed by a steady state bifurcation to a mixed parity state. This state loses stability in a further steady state bifurcation to a traveling wave which in turn loses stability at a Hopf bifurcation to a MTW. The MTW two-torus terminates in a collision with the two circles of pure parity states forming an attracting structurally stable heteroclinic cycle connecting them and their quarter-wavelength translates. In this regime the behavior would resemble that found in the numerical simulations, with higher modes kicking the system away from this cycle. Indeed this sequence of transitions echoes the results obtained by She and Nicolaenko for $k=8$. While it is likely that the $k=1$ scenario is relevant to these calculations because of the tendency towards an inverse cascade in these two-dimensional systems, it must be mentioned that the careful analysis of the $k=2$ case by Armbruster et al. [30] shows that while a heteroclinic cycle of the required type does indeed form it is not structurally stable. The case $k=4$ has also been studied [34] and a similar sequence of transitions found. However, despite much work a detailed understanding of the bursts in this system remains elusive, although as argued above simulations on $\{-\pi < x \leq \pi, -\pi/k < y \leq \pi/k\}$ could shed new light on the problem, cf. [35]. We mention here that closely related problems arise in convection in rotating straight channels [36] and in natural convection in a vertical slot [37]. In both of these cases the linear eigenfunctions are either even or odd with respect to a rotation by π.

2.4. Bursts in the Taylor-Couette system. The Taylor-Couette system consists of concentric cylinders enclosing a fluid-filled annulus. The cylinders can be rotated independently. In the counterrotating regime the

A related "punctuated Hamiltonian" approach to the evolution of two-dimensional turbulence has met with considerable success [18, 19]. For their description Newell et al. divide the instantaneous states of the flow into two categories, a turbulent soup (TS) characterized by weak coherence, and a singular (S) state characterized by strong coherence, and suppose that the TS and S states are generalized saddles in an appropriate phase space. Furthermore, they suppose that in the Hamiltonian limit the unstable manifold of TS (S) intersects transversally the stable manifold of S (TS). If the constant energy surfaces are noncompact (i.e. unbounded), the evolution of the Hamiltonian system may take the system into regions of phase space with very high ("infinite") velocities and small scales. These regions are identified with the S states and high dissipation. In such a scenario the strong dissipation events are therefore identified with excursions along heteroclinic connections to infinity. Perturbations to the system (such as the addition of dissipative processes) may prevent the trajectory from actually reaching infinity, but this underlying unperturbed structure implies that large excursions are still possible.

Newell et al. apply these ideas to the two-dimensional nonlinear Schrödinger equation (NLSE) with perturbations in the form of special driving and dissipative terms which act at large and small scales, respectively. Here S consists of "filament" solutions to the unperturbed NLSE which become singular in finite time and represent coherent structures which may occur at any position in the flow field. When the solution is near S a large portion of the energy is in small scales; for the perturbed equations the dissipative term then becomes important so that the filament solution is approached but collapses before it is reached. This leads to a spatially and temporally random occurrence of localized burst-like events for the perturbed equation. The rate of attraction at S is determined by the faster than exponential rate at which the filament becomes singular, while the rate of repulsion at S is governed by the dissipative process and hence is unrelated to the rate of attraction.

This bursting mechanism shares characteristics with that described in [20] in which solutions of a single complex Ginzburg-Landau equation with periodic boundary conditions undergo faster than exponential bursting due to a destabilizing nonlinearity and collapse due to strong nonlinear dispersion (see also [21]). A study of a generalization of Burger's equation modeling nonlocality effects suggests the presence of burst-like events through a similar scenario [22].

2.3. Bursts in the Kolmogorov flow. The Kolmogorov flow $\mathbf{u} = (k \sin ky, 0)$ is an exact solution of the two-dimensional incompressible Navier-Stokes equation with unidirectional forcing $\mathbf{f}$ at wavenumber k: $\mathbf{f} = (\nu k^3 \sin ky, 0)$. With increasing Reynolds number $Re \equiv \nu^{-1}$ this flow becomes unstable, and direct numerical simulation with 2π-periodic boundary conditions shows that for moderately high Reynolds numbers the result-is often characterized by intermittent bursting events involving low speed streamwise “streaks” of fluid. Specifically, let x_1, x_2, and x_3 be the streamwise, wall normal, and spanwise directions with associated velocity components $U + u_1, u_2$, and u_3, respectively; here $U(x_2)$ is the mean flow. In a “burst” the streak breaks up and low speed fluid moves upward away from the wall ($u_1 < 0, u_2 > 0$); this is followed by a “sweep” in which fast fluid moves downward towards the wall ($u_1 > 0, u_2 < 0$). After the burst/sweep cycle the streak reforms, often with a lateral spanwise shift.

A low-dimensional model of the burst/sweep cycle was developed by Aubry et al. [6]; further details and later references may be found in [7, 8]. To construct such a model the authors used a Karhunen-Loève decomposition of the experimental data to identify an energetically dominant empirical set of eigenfunctions, hereafter “modes”. The model was constructed by projecting the Navier-Stokes equation onto this basis and consists of a set of coupled ODEs for the amplitudes of these modes. The fixed points of these equations are to be associated with the presence of coherent structures. There are two types, related by half-wavelength translation. Numerical integration of the model reveals that these fixed points are typically unstable and that they are connected by a heteroclinic cycle. In such a cycle the trajectory visits the vicinity of one unstable fixed point to the other and back again. In the model of Aubry et al. this heteroclinic cycle is found to be structurally stable, i.e. it persists over a *range* of parameter values. This is a consequence of the O(2) symmetry of the equations inherited from periodic boundary conditions in the spanwise direction. Moreover, for the parameter values of interest this cycle is *attracting*, i.e., it attracts all nearby trajectories. Since the transition from one fixed point to the other corresponds to a spanwise translation by half a wavelength the recurrent excursions along such a heteroclinic cycle can be identified with the burst/sweep cycle described above. However, since this cycle is attracting, the time between successive bursts will increase as time progresses. This is not observed and Aubry et al. appeal to the presence of a random pressure term modeling the effect of the outer fluid layer to kick the trajectory from heteroclinic cycle. In the language of Busse [9] such a pressure term results in a *statistical* limit cycle, with the bursting events occurring randomly in time but with a well-defined mean rate. The resulting temporal distribution of the burst events is characterized by a strong exponential tail, matching experimental observations. Attracting structurally stable heteroclinic cycles occur in a number of problems of this type, i.e., mode interaction problems with O(2) symmetry [10]-[15].

2.2. Heteroclinic connections to infinity. A distinct mechanism, also involving heteroclinic connections, has been investigated by Newell et al. [16, 17] as a possible model for spatio-temporal intermittency in turbulent flow. The authors suggest that such systems may be viewed as nearly Hamiltonian except during periods of localized intense dissipation.

BURSTING MECHANISMS FOR HYDRODYNAMICAL SYSTEMS

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Abstract. Different mechanisms believed to be responsible for the generation of bursts in hydrodynamical systems are reviewed and a new mechanism capable of generating regular or irregular bursts of large dynamic range near threshold is described. The new mechanism is present in the interaction between oscillatory modes of odd and even parity in systems of large but finite aspect ratio, and provides an explanation for the bursting behavior observed in binary fluid convection by Sullivan and Ahlers.

1. Introduction. Bursts of activity, be they regular or irregular, are a common occurrence in physical and biological systems. In recent years several models of bursting behavior in hydrodynamical systems have been described using ideas from dynamical systems theory. In this article we provide a brief overview of these mechanisms and then describe a new mechanism [1] which provides an explanation for the bursting behavior observed in experiments on convection in ^{3}He/^{4}He mixtures [2]. This mechanism operates naturally in systems with broken D_4 symmetry undergoing a Hopf bifurcation from a trivial state. This symmetry may be present because of the geometry of the system under consideration (for example, the shape of the container) but also appears in large aspect ratio systems with reflection symmetry [3]. In either case bursting arises as a result of the nonlinear interaction between two nearly degenerate modes with different symmetries, one of which is subcritical and the other supercritical.

2. Mechanisms producing bursting. As detailed further below bursts come in many different forms, distinguished by their dynamic range, duration and recurrence properties. Particularly important for the purposes of the present article is the question of whether the observed bursts occur close to the threshold of a primary instability or whether they are found far from threshold. In the former case a dynamical systems approach is likely to be successful: in this regime the spatial structure usually resembles the eigenfunctions of the linear problem and it is likely that only a small number of degrees of freedom participate in the burst. In addition the equations governing the evolution of the instability are often highly symmetric [4] and these symmetries favor global bifurcations which serve as likely candidates for bursting mechanisms. In contrast, bursts found far from threshold usually involve many degrees of freedom but even here some progress is sometimes possible.

2.1. Bursts in the wall region of a turbulent boundary layer. The presence of coherent structures in a turbulent boundary layer is well established (see, e.g., [5]). The space-time evolution of these structures

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