Minimal Plane Couette Flow Turbulence: A Low-Dimensional, Uncoupled Model

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ABSTRACT

We model turbulent plane Couette flow for a Minimal Flow Unit (the smallest domain in which turbulence can be sustained) by expanding the velocity field as a sum of optimal modes calculated via the proper orthogonal decomposition from numerical data. Ordinary differential equations are obtained by Galerkin projection of the Navier-Stokes equations onto these modes. We consider an uncoupled 9 mode (16-dimensional) model, which provides evidence that the "backbone" for Minimal Flow Unit turbulence is a periodic orbit.

1 Introduction

In plane Couette flow (PCF), fluid is sheared between two infinite parallel plates moving at speed U_0 in opposite directions $\pm\mathbf{e}_x$; see Figure 1. The x, y, z-directions are defined to be the streamwise, wall normal, and spanwise directions, respectively. We nondimensionalize lengths in units of $d/2$ where d is the gap between the plates, velocities in units of U_0, time in units of $(d/2)/U_0$, and pressure in units of $U_0^2\rho$ where ρ is the fluid density. Laminar flow is then given by $\mathbf{U}_0 = y\mathbf{e}_x$, $-1 \leq y \leq 1$. The laminar state is linearly stable for all Reynolds numbers $Re = \frac{U_0 d}{2\nu}$ [1], where ν is the kinematic viscosity; however, both experiments and simulations exhibit sustained turbulence for sufficiently high Re and/or perturbation amplitudes (see, e.g., [2]). Writing $\mathbf{u} = (u_1, u_2, u_3)$, $\mathbf{x} = (x, y, z)$, the evolution equation for the perturbation $(\mathbf{u}(\mathbf{x}, t), p(\mathbf{x}, t))$ to laminar flow is

$$\frac{\partial}{\partial t}\mathbf{u} = -(\mathbf{u}\cdot\nabla)\mathbf{u} - y\frac{\partial}{\partial x}\mathbf{u} - u_2\mathbf{e}_x - \nabla p + \frac{1}{Re}\nabla^2\mathbf{u}. \quad (1)$$

The fluid is assumed to be incompressible and there are no-slip boundary conditions at the plates. Finally, the flow is assumed periodic in the streamwise and spanwise directions, with lengths $L_x \equiv 1.75\pi$ and $L_z \equiv 1.2\pi$, respectively. This corresponds to the Minimal Flow Unit (MFU), the smallest domain in which turbulence can be sustained for this system [3].

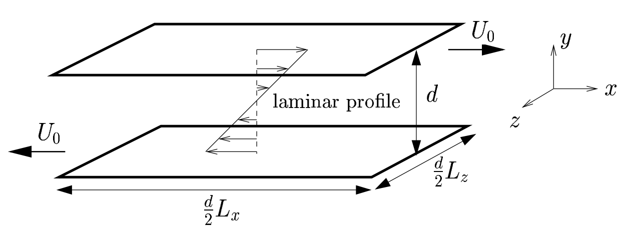

Figure 1: Geometry for plane Couette flow.

The nature of the weakly turbulent flow in the MFU was first described in [3]. Here the authors define the RMS modal velocities as

$$\mathrm{M}(n_x, n_z) = \left(\int_{-1}^{1} \left[\tilde{u}_1^2(n_x, y, n_z) + \tilde{u}_2^2(n_x, y, n_z)\tilde{u}_3^2(n_x, y, n_z) \right] dy \right)^{1/2}, \tag{2}$$

where the tildes represent Fourier mode amplitudes, and they discuss the temporal behavior of this quantity for various wavenumber pairs (n_x, n_z). They find that the RMS modal velocities for several modes shows almost periodic behavior and, in particular, that $\mathrm{M}(0,1)$ and $\mathrm{M}(1,0)$ are roughly of opposite phase: a peak in the former is often accompanied by a trough in the latter, and vice versa, as illustrated in Fig. 2(a) (cf. Fig. 3(a) in [3]). Fig. 2(b) (cf. Fig. 2 in [3]) shows midplane contours of the streamwise velocity at the times labeled on the $\mathrm{M}(0,1)$ curve in Fig. 2(a). At the time labeled ‘1’, the flow shows prominent streaks, that is, streamwise-coherent structures with variation of the streamwise velocity with respect to spanwise position. The streaks have broken down by the time labeled ‘5’, then they regenerate and the process begins anew.

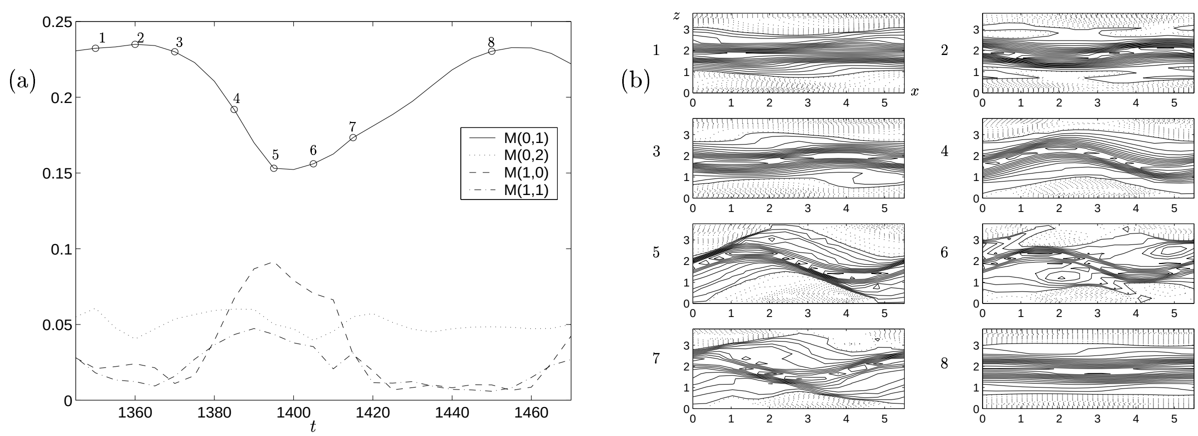

Figure 2: (a) RMS modal velocities for several wavenumbers, and (b) midplane streamwise-velocity contours for one representative cycle from the direct numerical simulations.

2 The Model

To model turbulent PCF, we perform a proper orthogonal decomposition (POD) on data from direct numerical simulations (DNS) of (1) at $Re = 400$. This identifies an energetically dominant set of empirical eigenmodes (“POD modes”) from the data. We then construct models by Galerkin projection of (1) onto finite-dimensional subspaces spanned by the dominant modes, yielding ordinary differential equations for the evolution of the modal amplitudes; see [4] for details and references on this procedure. We have previously used it to derive low-dimensional models for PCF turbulence at $Re = 400$ for a moderate aspect-ratio domain with $L_x = 4\pi$, $L_z = 2\pi$ [5], and a coupled 6 mode (11-dimensional) model for MFU PCF turbulence at $Re = 400$ [6, 7]. Here we consider an *uncoupled* 9 mode (16-dimensional) model, as explained below. Note that, like [5], here we do not model losses to neglected modes. On the other hand, unlike [5, 6], we *do* subtract off the time-averaged mean flow from our DNS snapshots before finding the POD modes; this is found to give better agreement between the behavior of the model and the DNS [7].

Respecting translation invariance in x and z, we take POD modes of the form

$$\mathbf{\Phi}^{(n)}_{n_x n_z}(\mathbf{x}) = \frac{\phi^{(n)}_{n_x n_z}(y)}{\sqrt{L_x L_z}} \exp\left(2\pi i \left(\frac{n_x x}{L_x} + \frac{n_z z}{L_z}\right)\right), \tag{3}$$

and, following the suggestion of [8], consider the decomposition of each mode into two orthogonal parts $\mathbf{\Phi}^{(n)}_{n_x n_z}(\mathbf{x}) = \mathbf{\Phi}^{(n)[1]}_{n_x n_z}(\mathbf{x}) + \mathbf{\Phi}^{(n)[2]}_{n_x n_z}(\mathbf{x})$. Here we set $\mathbf{\Phi}^{(n)[1]}_{n_x n_z}(\mathbf{x}) = P\mathbf{\Phi}^{(n)}_{n_x n_z}(\mathbf{x})$ and $\mathbf{\Phi}^{(n)[2]}_{n_x n_z}(\mathbf{x}) = (I-P)\mathbf{\Phi}^{(n)}_{n_x n_z}(\mathbf{x})$, where the projection matrix $P = pp^T/(p^T p)$ and $p = [-2\pi n_z/L_z, 0, 2\pi n_x/L_x]^T$. Uncoupling the POD modes in this manner leads to models which properly uncouple the kinematically independent degrees of freedom of the Navier-Stokes equations [8] (cf. [9]). The perturbation velocity field $\mathbf{u}$ in terms of the uncoupled POD modes is then

$$\mathbf{u}(\mathbf{x},t) = \begin{pmatrix} U_m(y) \\ 0 \\ 0 \end{pmatrix} + \sum_{n=1}^{\infty} \sum_{n_x=-\infty}^{\infty} \sum_{n_z=-\infty}^{\infty} \sum_{m=1}^{2} b^{(n)[m]}_{n_x n_z}(t) \mathbf{\Phi}^{(n)[m]}_{n_x n_z} \tag{4}$$

where $(U_m(y), 0, 0)^T$ is the time-averaged mean flow. Our present model derives from a Galerkin projection of (1) onto (4) truncated to include only the (uncoupled) $(1,0,0)$, $(1,0,1)$, $(1,0,2)$, $(1,0,3)$, $(1,1,0)$, $(1,1,\pm1)$, and $(1,1,\pm2)$ modes. Using 4000 snapshots of the numerical data (expanded to $4 \times 4000 = 16000$ snapshots by symmetry operations, see [4, 5]), these modes contain 89.37% of the average total energy.

Figure 3 shows the analogue to Figure 2 for our model at $Re = 400$. As was found for the model in [6], the present model captures the streak breakdown and regeneration process as an attracting periodic orbit with reasonable period of M(0,1) and M(1,0): 91.2 nondimensional time units, versus 80-100 for the DNS. These RMS modal velocities also approximately have the desired phase relationship. Unlike the model considered in [6], the present model also predicts that a fixed point which corresponds to the laminar state remains stable for all Re, and that many steady solutions arise through saddle-node bifurcations, as is known to occur for PCF [10, 11]. See [7] for more detail on these models and their behavior.

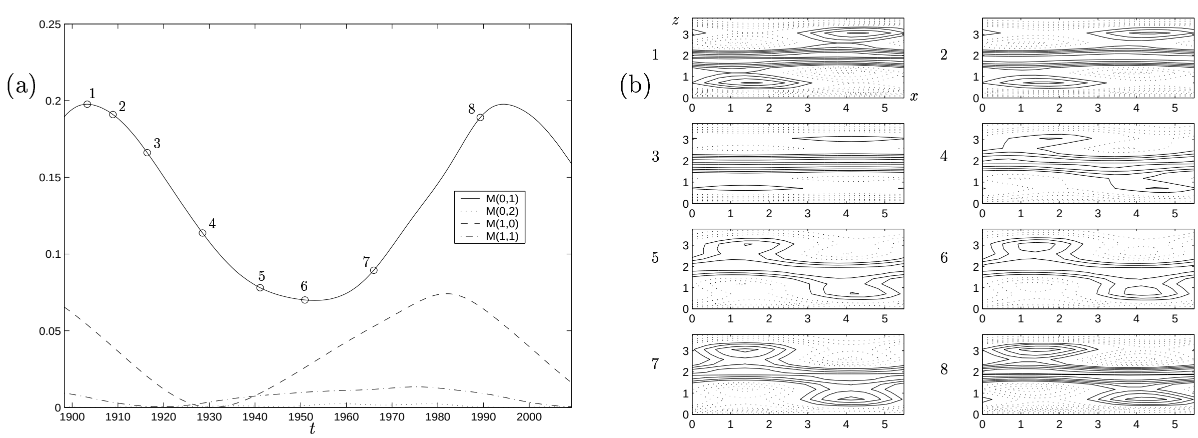

Figure 3: (a) RMS modal velocities for several wavenumbers, and (b) midplane streamwise-velocity contours for one period for the model at $Re = 400$.

3 Conclusion

We have modeled turbulent plane Couette flow for a Minimal Flow Unit (the smallest domain in which turbulence can be sustained) by expanding the velocity field as a sum of optimal modes calculated via the proper orthogonal decomposition from numerical data. Ordinary differential equations were obtained by Galerkin projection of the Navier-Stokes equations onto these modes. We found that an uncoupled 9 mode (16-dimensional) model nicely captures the streak breakdown and regeneration process as a periodic orbit with reasonable period and phase relationships between the RMS modal velocities. These results provide a rationale for dimension reduction, produce ordinary differential equations that behave consistently with other models for Minimal Flow Unit plane Couette flow turbulence [6, 7], and may also lead to better analytical understanding of instability mechanisms.

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