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Nonlinear Hybrid Control of Phase Models for Coupled Oscillators

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***Abstract*— We present a new approach to the problem of desynchronization of a population of all-to-all coupled oscillators. Motivated by Deep Brain Stimulation treatment of Parkinson's Disease, the objective is to break this synchrony in the fullest measure by means of a control input only applied to one of the oscillators. Specifically, nonlinear hybrid control is proposed as a novel method for robust global asymptotic stabilization of the splay state. The problem setup is presented in a general way, and a simple example is solved that gives an idea of how this method might be applied in practice.**

I. INTRODUCTION

Populations of periodically firing neurons in the brain are often modeled as networks of coupled oscillators, e.g. [1], [2]. Pathological synchrony of these neurons in the motor control region of the brain sometimes results in Parkinson's disease, for which, Deep Brain Stimulation (DBS) has been proven to be an effective treatment. In DBS, an electrical stimulus is injected into the brain to desynchronize the firing. Recently, the use of phase models has become more common in studies related to controlling neurons [3], [4], [5], [6]. However, these studies have primarily been on a single-neuron level [4], [6], or else, at the population level, they have allowed multiple control inputs to the system [5]. In this study, nonlinear hybrid control [7] is proposed as a new approach to controlling a population of neurons with only a single control input. However, at this early stage, we have made two simplifying assumptions: observability of phases of all neurons and simple additive control.

II. MODEL SETUP

We consider a phase model for a network of N coupled oscillators, subject to one control input that, without loss of generality is applied to the N^{th} oscillator in the network:

$$\dot{\theta}_i = \omega + \sum_{j=1}^{N} \alpha_{ij} f(\theta_j - \theta_i) + \delta_{iN} u, \quad \theta_i \in [0, 2\pi), \qquad (1)$$

for $i = 1, 2, ..., N$. θ_i is the phase of oscillator i, ω is the oscillators' natural frequency, α_{ij} is the coupling strength from oscillator j to oscillator i, $f(\cdot)$ is the 2π-periodic coupling function, δ is the Kronecker delta function, and u is the control input. By convention, neuron i fires when $\theta_i = 0$.

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We note that all oscillators are assumed to have identical ω, and that the functional form of the coupling between any pair of neurons is identical, although the strength of such coupling can differ. We can simplify this system by defining $\psi_i = \theta_N - \theta_i$ for $i = 1, 2, .., N-1$, to obtain the $N-1$ dimensional system $\dot{\psi}_i = \Omega_i(\psi) + u$ where $\psi = (\psi_1, \psi_2, ..., \psi_{N-1})^T$ is the vector of phase differences and $\Omega_i(\psi)$ is the resulting uncontrolled vector field. Our objective here is to not only break the in-phase synchrony between the oscillators, but to break it to the fullest measure and stabilize the *splay* state, for which the oscillators' phases are distributed evenly on the unit phase circle, every two neighboring oscillators being $\frac{2\pi}{N}$ radians apart. That is, we want to stabilize $\psi_i = (N-i)\frac{2\pi}{N}$. Performing a coordinate transformation to move the splay state to the origin, we get the following ξ system:

$$\dot{\xi}_i = \Omega_i(\xi) + u, \quad i = 1, 2, ..., N-1, \qquad (2)$$

where $\xi = (\xi_1, \xi_2, ..., \xi_{N-1})^T$ and $\Omega_i(\xi)$ is the uncontrolled vector field. Now, if we apply the coordinate transformation $x_{2i-1} = \cos(\xi_i)$ and $x_{2i} = \sin(\xi_i)$, we get the following $2(N-1)$ dimensional system:

$$\dot{x}_{2i-1} = -(\Omega_i(\xi) + u)x_{2i} \qquad \dot{x}_{2i} = (\Omega_i(\xi) + u)x_{2i-1}, \qquad (3)$$

with $N-1$ constraints: $x_{2i-1}^2 + x_{2i}^2 = 1$, for $i = 1, 2, \cdots, N-1$. Stabilizing $\xi_i = 0$ in (2) corresponds to stabilizing $(x_{2i-1}, x_{2i}) = (1, 0)$ in (3). This is in fact, a problem of stabilization of $N-1$ points on a circle.

To investigate the control strategy for this system, we consider the following example. We emphasize that the control strategy taken in this example should not be viewed as a definite approach towards controlling such systems, but it is suggestive that there might be a way to control networks of coupled oscillators with only one input.

III. EXAMPLE AND CONTROL STRATEGY

We consider $N = 3$, the coupling function $f(x) = \sin(3x)$, and symmetric coupling with $\alpha_{ij} = \alpha_{ji}$. When one does the aforementioned coordinate transformations, one obtains (3) with $i = 1, 2$, where here, $x_1 = \cos(\xi_1)$, $x_2 = \sin(\xi_1)$, $x_3 = \cos(\xi_2)$, and $x_4 = \sin(\xi_2)$. The goal is to stabilize the $\xi_i = 0$, or equivalently $X = (1, 0, 1, 0)^T$ for this system. We will apply a series of different control laws to accomplish our goal, hence the name hybrid control.

The control strategy for this example is as follows. We first restrict our attention to one of the oscillators, say ξ_1. We apply the same approach as in Example 34 of [7] for robust global stabilization of a point on a circle. We make sure that we apply a control that would steer this oscillator to $(x_1, x_2) = (1, 0)$ regardless of what the other oscillator is doing. To this end, we introduce $u = -\Omega_1(\xi) - x_2$ as the main control law that achieves *almost* global asymptotic stabilization for $(x_1, x_2) = (1, 0)$ (by almost, we mean that there is a measure zero set for which this control is not effective). To see this, we look at the first two equations in (3). If we replace this u in these equations we get:

$$\dot{x}_1 = x_2^2, \qquad \dot{x}_2 = -x_1 x_2. \tag{4}$$

We can prove asymptotic stabilization of $(x_1, x_2) = (1, 0)$ or equivalently, $\xi_1 = 0$, using the Lyapunov function $V_1(x_1, x_2) = \frac{1}{2}(x_1 - 1)^2 + \frac{1}{2}x_2^2$. The derivative of this Lyapunov function along the trajectories of the system is $\dot{V}_1 = -x_2^2 \leq 0$. In accordance with Barbashin-Krasovskii-LaSalle invariance principle, we argue that $\dot{V} \equiv 0$ implies $x_2 \equiv 0$, which considering (4) and the fact that $x_1^2 + x_2^2 = 1$, further implies that $x_1 = \pm 1$. Therefore, $u = -\Omega_1(\xi) - x_2$ asymptotically stabilizes $(x_1, x_2) = (1, 0)$ for $\forall (x_1, x_2) \neq (-1, 0)$. $(x_1, x_2) = (-1, 0)$ is a fixed point for the closed loop system (4), i.e., at this point we need another control, namely an *auxiliary* control to steer the system out of this state before switching to the main controller. This auxiliary control is taken to be $u = -\Omega_1(\xi) - x_1$, resulting in

$$\dot{x}_1 = x_1 x_2, \qquad \dot{x}_2 = -x_1^2. \tag{5}$$

The time derivative of the Lyapunov function $V_2(x_1, x_2) = \frac{1}{2}x_1^2 + \frac{1}{2}(x_2 + 1)^2$ along the trajectories of (5) is $\dot{V}_2 = -x_1^2 \leq 0$, implying local asymptotic stability for $(x_1, x_2) = (0, -1)$. This implies that the state is pushed down on the circle and away from $(x_1, x_2) = (-1, 0)$. The domain of applicability of this auxiliary control is taken to be a small neighborhood of $(-1, 0)$, and the domain of applicability of the main control is taken to be everywhere on the circle except for a smaller neighborhood of $(-1, 0)$. There is a hysteresis region between the domains of applicability of the two controls to avoid chattering in the system. The combination of the two controllers globally asymptotically stabilizes $(1, 0)$ for the first oscillator. Fig. 1 shows the phase plane of the system in the ξ coordinates under $u = -\Omega_1(\xi) - x_2$.

Now we turn our attention to the second oscillator ξ_2. We see that for this example, depending on the initial conditions, (x_3, x_4) asymptotically approach $(1, 0)$ or $(-0.5, \pm\sqrt{3}/2)$, which is equivalent to $\xi_2 = 0, \pm 2\pi/3$. If not at $(1, 0)$, we apply $u = -\Omega_2(\xi) - x_3$ to (3) to force the second oscillator away from these points and towards $(0, -1)$. This control ensures asymptotic stability of $(x_3, x_4) = (0, -1)$ by similar Lyapunov function arguments as before. However, upon application of this control, considering (2) one can verify that $\dot{\xi}_1$ becomes strictly positive, resulting in continuous CCW rotation of ξ_1 on the circle. With ξ_2 asymptotically close to $3\pi/2$ and ξ_1 continuously increasing, in finite time, the overall state of the system falls in the region of $\pi < \xi_1 < 3\pi/2$ and $\xi_2 \approx 3\pi/2$ where the main control $u = -\Omega_1(\xi) - x_2$ could be switched on again to steer both ξ_1 and ξ_2 to the desired location, as can be seen from Fig. 1. This method of control robustly and globally stabilizes the splay state for our network of three oscillators. The results are shown for the in-phase initial condition in terms of the original θ variables introduced in (1). Fig. 2 shows the phase differences and the overall control law and the instances in time that the control law has been changed.

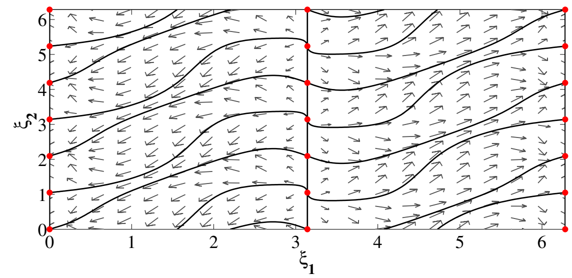

Fig. 1. Phase space of $\xi_1 - \xi_2$ system under $u(X) = -\Omega_1(X) - x_2$. We stabilize $(\xi_1, \xi_2) = (0, 0)$(or equivalently, $(\xi_1, \xi_2) = (2\pi, 2\pi)$).

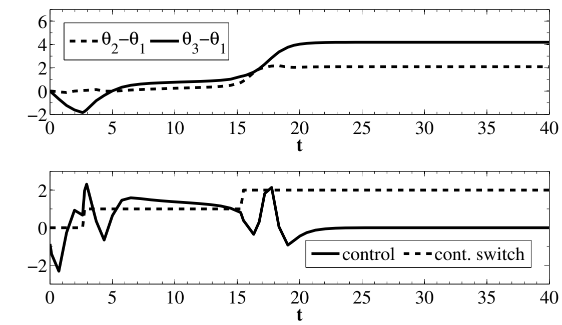

Fig. 2. Simulation result for the in-phase initial condition, (top) θ_2 and θ_3 relative to θ_1 are shown to converge to $2\pi/3$ and $4\pi/3$, respectively; (bottom) the control input and the number of control switches are shown.

IV. FUTURE WORK

The example presented here suggests that there may be potential for this method of control for more general networks of coupled oscillators. However, one would need to address the problem for arbitrary coupling functions $f(\cdot)$, coupling strengths α_{ij}, and oscillator numbers N, and it would be desireable to relax the observability condition. There is also potential for research in optimizing the hybrid control method based on the total input energy or the total time.

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