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ON THE DYNAMICS OF COUPLED PARAMETRICALLY FORCED OSCILLATORS

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ABSTRACT

It is well known that an autonomous dynamical system can have a stable periodic orbit, arising for example through a Hopf bifurcation. When a collection of such oscillators is coupled together, the system can display a number of phase-locked solutions which can be understood in the weak coupling limit by using a phase model. It is also well known that a stable periodic orbit can be found for a parametrically forced dynamical system, with the phase of the periodic orbit being locked to the forcing. Here we discuss the periodic solutions which occur for a collection of such parametrically forced oscillators that are weakly coupled together.

INTRODUCTION

The scientific study of coupled oscillators started with Christian Huygens' observations in the seventeenth century of mutual synchronization of pendulum clocks connected by a beam [1,2]. More recently, it has been recognized that mutual synchronization of coupled oscillators - the adjustment of rhythms of oscillating objects due to their weak interactions - occurs in many biological systems, including neurons during epileptic seizures [3] and pacemaker cells in the human heart [4]. Coupled oscillators have also been studied in detail for technological systems, such as arrays of lasers and superconducting Josephson junctions: see [5], [6], and [7], a recent popular book on the topic, for many biological and technological examples of synchronization for coupled oscillators.

We classify as *autonomous oscillators* those for which the stable oscillations occur for an autonomous dynamical system, that is one for which there are no explicit time-dependent terms in the evolution equation. For example, the oscillations might arise through a Hopf bifurcation, as for the microelectromechanical systems (MEMS) oscillators considered in [8,9]. In the limit of weak coupling, it is possible to reduce the dynamics of coupled autonomous oscillators to a phase model, with a single variable describing the phase of each oscillator with respect to some reference state (see, e.g., [10–13]). This typically leads to models for which the dynamics depend only on the phase differences between different oscillators. It is possible to show that several types of phase-locked solutions, for which the phase of all oscillators increases at the same constant rate, are guaranteed to exist in the weak coupling limit for *any* generic coupling function when the coupling topology has appropriate symmetry properties [14–17]; for the case of identical all-to-all coupling for N oscillators, these are

- in phase solution: all N oscillators have the same phase
- two-block solutions: there are two blocks of oscillators, one in which p oscillators share the same phase, and one in which $N-p$ oscillators share the same phase
- rotating block solutions: for $N = mk$, there are m blocks with k oscillators in each block sharing the same phase, with neighboring blocks differing in phase by $2\pi/m$
- double rotating block solutions: for $N = m(k_1 + k_2)$, there are two rotating block solutions, one with m blocks with k_1 oscillators in each block sharing the same phase and with neighboring blocks differing in phase by $2\pi/m$, another with m blocks with k_2 oscillators in each block sharing the same phase and with neighboring blocks differing in phase by $2\pi/m$, where there is a phase difference $0 < \phi < 2\pi/m$ between a block with k_1 oscillators and the closest phase-advanced block with k_2 oscillators.

On the other hand, we classify as *non-autonomous oscillators* those for which the stable oscillations only occur for a non-autonomous dynamical system, that is one for which there are explicit time-dependent terms such as time-periodic forcing. We will focus on parametrically forced oscillators, which are non-autonomous oscillators for which the forcing enters as a time-varying system parameter. Coupled parametrically forced oscillators arise in MEMS [18–20] and other application areas [21–24], but have not received as much theoretical research attention as coupled autonomous oscillator systems. This paper represents the first steps in developing a comprehensive theory of the dynamics of general weakly coupled non-autonomous oscillators, in the spirit of the theory of general weakly coupled autonomous oscillators described in [14, 17]. We hope that such a theory will ultimately lead to novel sensing mechanisms using MEMS devices; for simplicity, here we will consider a model system which represents only a caricature of such devices.

Specifically, in this paper we describe interesting synchronization phenomena that are possible for coupled parametrically forced oscillators. For example, consider two *uncoupled* oscillators whose response is at half the frequency of the driving voltage, as is common for MEMS devices [25]. Both oscillators could identically lock to the forcing, or they could lock one forcing period apart - both situations are allowable due to a discrete time-translation symmetry for the problem. We will show that different combinations of these states will persist if the oscillators are now weakly *coupled*, with stability inherited from the stability properties of the periodic orbits which exist for the uncoupled system.

We first consider the dynamics of a specific single parametrically forced oscillator. We then consider two uncoupled parametrically forced oscillators, identifying different periodic states for such systems. Next, we show that provided the periodic orbits for the uncoupled system are hyperbolic, there will be periodic orbits for the coupled system close to the periodic states identified for the uncoupled system. This is then demonstrated through numerical bifurcation analysis. We finally describe how these results can be generalized to N coupled parametrically forced oscillators.

A PARAMETRICALLY FORCED OSCILLATOR

Consider the equation for a damped, parametrically forced oscillator

$$\ddot{x} + b\dot{x} + x + x^3 = xF\cos(\omega_f t). \quad (1)$$

Here the term $b\dot{x}$ represents damping (we assume $b > 0$), the term $x + x^3$ represents a nonlinear restoring force, and the term $xF\cos(\omega_f t)$ represents parametric excitation which can be viewed as a time-periodic modulation of the linear part of the restoring force. For this system, if $F = 0$ then $x \to 0$ as $t \to \infty$, as follows. Letting

$$V(x,\dot{x}) = \frac{1}{2}x^2 + \frac{1}{4}x^4 + \frac{1}{2}\dot{x}^2, \quad (2)$$

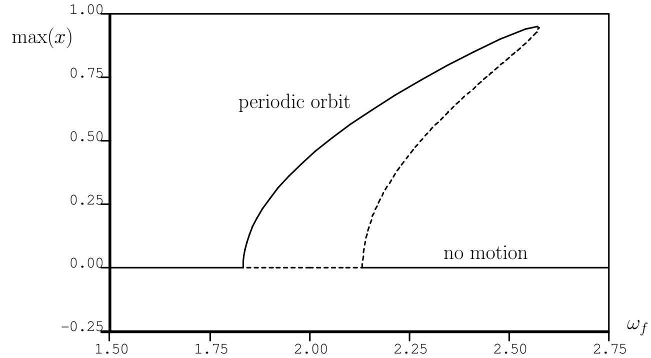

Figure 1. BIFURCATION DIAGRAM FOR FIXED $b = 0.2$ and $F = 0.5$. SOLID (RESP., DASHED) LINES INDICATE STABLE (RESP., UNSTABLE) SOLUTIONS.

we find that $\frac{dV}{dt} = -b\dot{x}^2 \leq 0$, with equality only if $\dot{x} = 0$. The only point in phase space which starts in the set of points for which $V(x,\dot{x}) = 0$ and remains in this set for all time is $(x,\dot{x}) = (0,0)$; by the LaSalle Invariance Principle [26], all trajectories thus approach this point as $t \to \infty$.

For appropriate F and ω_f, the system has a periodic response. Indeed, treating ω_f as a bifurcation parameter for fixed F and b, we obtain the bifurcation diagram shown in Fig. 1. (This and other numerical bifurcation analysis was done using AUTO [27].) The "no motion" state is characterized by $x = \dot{x} = 0$ for all time, and will also be referred to as the 0 solution; it exists for all ω_f, being unstable near $\omega_f = 2$ and stable otherwise for the range shown. It loses stability in a bifurcation to a periodic orbit, with the periodic orbit branch turning around in a saddlenode bifurcation so that there is a region of bistability between the periodic orbit and the no motion state. Such a bifurcation structure is common for MEMS devices, see e.g. [28, 29]. Figure 2 indicates the types of dynamics which occur in different parameter ranges, with the "parabola" corresponding to the loss of stability of the no motion state, and the "straight line" corresponding to the saddlenode bifurcation of the periodic orbit branch. As shown in Fig. 3, the response of this periodic orbit is at half the frequency of the forcing, as is common for parametrically forced oscillators [30]. We note that an equally valid periodic orbit for this forcing is shown in Fig. 4, which is shifted by one period of the forcing from the solution shown in Fig. 3; clearly these solutions are related by a time-translation symmetry. These symmetry-related solutions will be crucial for understanding the different possible solutions when such oscillators are weakly coupled. We will find it convenient to distinguish the solutions shown in Fig. 3 and 4 by referring to one of them as the A solution and the other as the B solution. Notice that the A and B solutions have the same max(x) value, and the periodic orbit branch in Fig. 1 corresponds to both (symmetry-related) solutions.

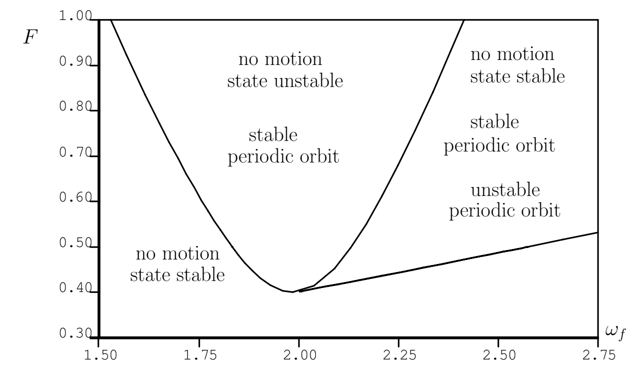

Figure 2. EXISTENCE AND STABILITY OF SOLUTIONS FOR DIFFERENT REGIONS OF PARAMETER SPACE FOR $b = 0.2$, WITH BIFURCATION SETS SHOWN AS LINES.

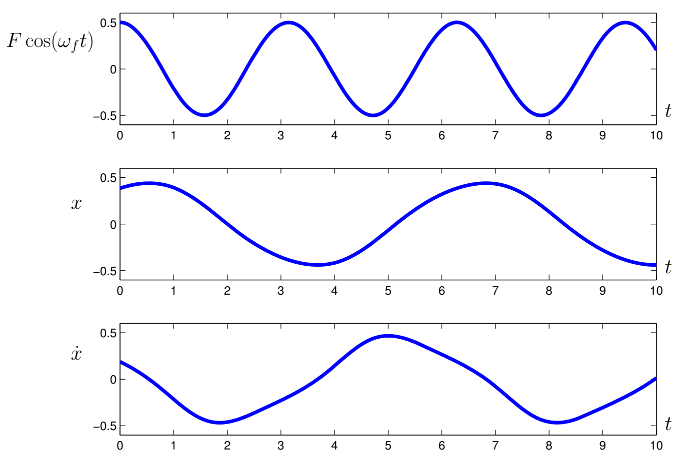

Figure 3. STABLE PERIODIC ORBIT FOR $b = 0.2, F = 0.5, \omega_f = 2$. THE RESPONSE IS AT HALF THE FREQUENCY OF THE FORCING. WE WILL REFER TO THIS AS THE A SOLUTION.

COUPLED PARAMETRICALLY FORCED OSCILLATORS

Now consider $N = 2$ parametrically forced oscillators which are coupled linearly:

$$\ddot{x}_1 + b\dot{x}_1 + x_1 + x_1^3 = x_1 F\cos(\omega_f t) + c(x_2 - x_1), \quad (3)$$
$$\ddot{x}_2 + b\dot{x}_2 + x_2 + x_2^3 = x_2 F\cos(\omega_f t) + c(x_1 - x_2), \quad (4)$$

where x_i is the position of the i^{th} oscillator, $i = 1, 2$.

$N = 2$ Uncoupled Oscillators

If $c = 0$, these are independent parametrically forced oscillators. Thus, for $b = 0.2, F = 0.5$, and $\omega_f = 2$, each oscillator could be in a stable periodic state given by the A or B solutions;

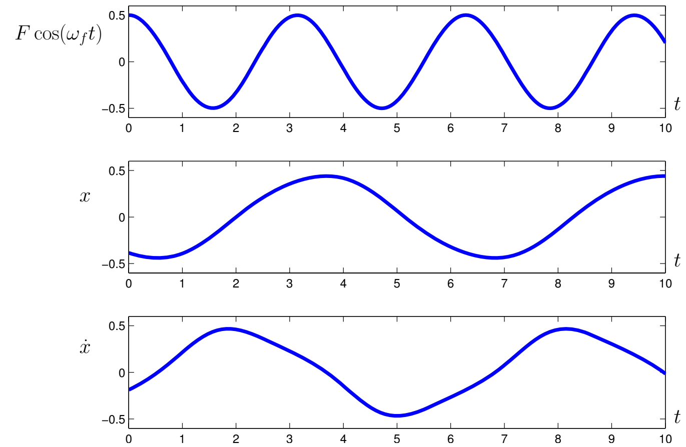

Figure 4. STABLE PERIODIC ORBIT FOR $b = 0.2, F = 0.5, \omega_f = 2$. THIS IS SHIFTED BY ONE PERIOD OF THE FORCING FROM THE SOLUTION SHOWN IN FIG. 3. WE WILL REFER TO THIS AS THE B SOLUTION.

each oscillator also has an unstable no motion state given by the 0 solution. The periodic solutions for the uncoupled two oscillator system are thus

in phase: $A \cdot A$, $B \cdot B$
out of phase: $A \cdot B$, $B \cdot A$
large-small: $A \cdot 0$, $B \cdot 0$, $0 \cdot A$, $0 \cdot B$
no motion: $0 \cdot 0$.

Here the first symbol characterizes the state of the first oscillator, and the second symbol characterizes the state of the second oscillator. *In phase* solutions have both oscillators responding identically to the forcing, while *out of phase* solutions correspond to each oscillator undergoing an oscillation which is shifted by one period of the forcing relative to the other. *Large-small solutions* have one oscillator undergoing oscillations locked to the forcing while the other oscillator is stationary. The name comes from the result that when weak coupling is introduced, the former oscillator will undergo relatively *large* oscillations, while the latter will undergo relatively *small* oscillations. For the *no motion* state, both oscillators are stationary. The distinct solutions within a given class (in phase, out of phase, or large-small) are related by symmetry.

Periodic Orbits for Weakly Coupled Oscillators

When $c \neq 0$ but is small, we expect analogous solutions to exist, as follows. We rewrite Eqs. (3,4) as

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + c\mathbf{g}(\mathbf{x}), \quad (5)$$

where

$$\mathbf{x} = (x_1, \dot{x}_1, x_2, \dot{x}_2), \quad (6)$$

and $\mathbf{f}(\mathbf{x})$ captures the terms which are independent of c and $\mathbf{g}(\mathbf{x})$ captures the coupling terms. Let $T = 2\pi/\omega_f$ be the period of the forcing. We define P_c to be the time-$2T$ map, that is the map which takes an initial condition (the state at $t = 0$) to the state obtained by evolving for a time equal to twice the period of the forcing. Now, let

$$h(\mathbf{x}, c) = P_c(\mathbf{x}) - \mathbf{x}, \tag{7}$$

and let $\mathbf{q}_0$ be a point on one of the periodic solutions of the uncoupled problem, for example the $A \cdot A$ solution. We see that

$$h(\mathbf{q}_0, 0) = 0. \tag{8}$$

The implicit function theorem (see Appendix) then implies that, provided the matrix $D_{\mathbf{x}}h(\mathbf{q}_0, 0)$ is invertible, there is a unique solution $\mathbf{q}(c)$ close to $\mathbf{q}_0$, for any sufficiently small c, such that $h(\mathbf{q}(c), c) = 0$. This implies that

$$P_c(\mathbf{q}(c)) = \mathbf{q}(c), \tag{9}$$

that is, $\mathbf{q}(c) \approx \mathbf{q}_0$ is a fixed point of the time-$2T$ map, which means that it is a point on a periodic orbit with period $2T$ which is close to a periodic orbit of the uncoupled system. (A related argument is used to prove part (ii) of Theorem 4.1.1 of [31].)

It is instructive to consider an alternative, but equivalent argument. We know that $\mathbf{q}_0$ is a fixed point of P_0, that is

$$P_0(\mathbf{q}_0) = \mathbf{q}_0. \tag{10}$$

We will determine, to leading order in c, the condition which must be met for $\mathbf{q}(c)$ to be a fixed point of P_c. Consider the asymptotic expansions

$$\mathbf{q}(c) = \mathbf{q}_0 + c\mathbf{q}_1 + \cdots \tag{11}$$
$$P_c(\mathbf{x}) = P_0(\mathbf{x}) + cp_1(\mathbf{x}) + \cdots. \tag{12}$$

Setting $P_c(\mathbf{q}(c)) = \mathbf{q}(c)$, we obtain

$$\begin{aligned}\mathbf{q}_0 + c\mathbf{q}_1 + \cdots &= P_c(\mathbf{q}_0 + c\mathbf{q}_1 + \cdots) \\ &= P_0(\mathbf{q}_0 + c\mathbf{q}_1 + \cdots) + p_1(\mathbf{q}_0 + c\mathbf{q}_1 + \cdots) \\ &= P_0(\mathbf{q}_0) + cDP_0(\mathbf{q}_0)\mathbf{q}_1 + cp_1(\mathbf{q}_0) + \cdots.\end{aligned}$$

This is valid at $O(c^0)$ from Eq. (10). At $O(c^1)$, we need

$$\mathbf{q}_1 = DP_0(\mathbf{q}_0)\mathbf{q}_1 + p_1(\mathbf{q}_0). \tag{13}$$

Solving for $\mathbf{q}_1$,

$$\mathbf{q}_1 = [Id - DP_0(\mathbf{q}_0)]^{-1} p_1(\mathbf{q}_0), \tag{14}$$

where Id is the identity matrix. In order to solve for $\mathbf{q}_1$, it is necessary that $[Id - DP_0(\mathbf{q}_0)]$ is invertible. This is equivalent to the condition above for the implicit function theorem to hold that $D_{\mathbf{x}}h(\mathbf{q}_0, 0)$ is invertible.

We now show that this matrix is invertible provided the periodic orbit for the uncoupled system is hyperbolic. Suppose that v is an eigenvector of $DP_0(\mathbf{q}_0)$ with eigenvalue λ, so that

$$[DP_0(\mathbf{q}_0)]v = \lambda v. \tag{15}$$

Then

$$(Id - [DP_0(\mathbf{q}_0)])v = (1 - \lambda)v. \tag{16}$$

Thus, the matrix $(Id - [DP_0(\mathbf{q}_0)])$ only has a zero eigenvalues if $\lambda = 1$. But the eigenvalues of $DP_0(\mathbf{q}_0)$ give the stability of the periodic orbit for the uncoupled problem; in particular, if it is a hyperbolic periodic orbit, none of the eigenvalues are on the unit circle. The hyperbolicity condition only needs to be checked for a single uncoupled oscillator, since we are assuming that the oscillators are identical.

Summarizing, provided the periodic orbit for the uncoupled system is hyperbolic, there will be a nearby periodic orbit for the system with sufficiently small coupling.

Furthermore, we expect that since the $c \to 0$ system limits to the $c = 0$ system, the periodic orbit for the weakly coupled system will "inherit" the stability properties from the periodic orbit for the uncoupled system. This follows from the continuity of the Poincaré map with respect to c, giving

$$\lim_{c \to 0} DP_c(\mathbf{q}(c)) = DP_0(\mathbf{q}_0). \tag{17}$$

This implies that the eigenvalues corresponding to the stability of the $\mathbf{q}(c)$ periodic orbit for the coupled system tend toward the eigenvalues corresponding to the stability of the $\mathbf{q}_0$ periodic orbit for the uncoupled system.

$N = 2$ Coupled Oscillators

These general results are illustrated for Eqs. (3,4) for $b = 0.2, F = 0.5, \omega_f = 2$ in Fig. 5, which shows that for small $c > 0$, periodic orbits of the expected stability type exist which are close to the periodic orbits for the uncoupled system. Indeed, for the uncoupled system the A and B solutions are stable and the 0 solution is unstable (see Fig. 1); thus, the $A \cdot A$ and $A \cdot B$ solutions (and their symmetry-related counterparts $B \cdot B$ and $B \cdot A$, respectively) are expected to be stable for small $|c|$, while the others are

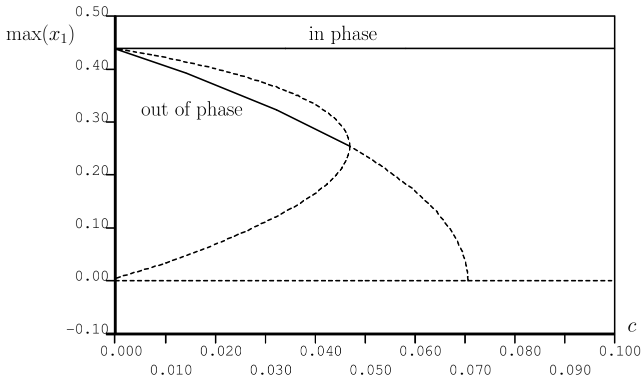

Figure 5. BIFURCATION DIAGRAM FOR FIXED $b = 0.2, F = 0.5, \omega_f = 2$ WITH THE COUPLING STRENGTH c TREATED AS A BIFURCATION PARAMETER. SOLID (RESP., DASHED) LINES INDICATE STABLE (RESP., UNSTABLE) SOLUTIONS. FOR SMALL (POSITIVE) c, FROM TOP TO BOTTOM THE SOLUTIONS ON THE BRANCHES APPROACH THE $A \cdot A$, $A \cdot 0$, $A \cdot B$, $0 \cdot A$, and $0 \cdot 0$ AS $c \to 0$. NOTE, FOR EXAMPLE, THAT THE $A \cdot A$ and $B \cdot B$ SOLUTIONS HAVE THE SAME $\max(x_1)$ VALUE, AND THUS APPEAR TO BE ON THE SAME BRANCH IN THIS PROJECTION. A SIMILAR COINCIDENCE BETWEEN SYMMETRY-RELATED SOLUTIONS OCCURS FOR ALL OTHER BRANCHES.

expected to be unstable. Figure 5 shows that that as c increases, the out of phase $A \cdot B$ solution loses stability. This illustrates that our arguments above are only valid for small $|c|$. We note that, in this figure, the fact that the branches come together at $c = 0$ is an artifact of the projection. The solutions are actually separated in phase space: even though they share the same value for $\max(x_1)$, the second oscillator has different behavior. Therefore, the uniqueness property from the implicit function theorem argument is not violated.

For $b = 0.2, F = 0.5, \omega_f = 2.25$, Fig. 1 shows that the no motion state is stable, and that there are stable periodic orbits (which are analogues of the A and B solutions discussed above) and unstable periodic orbits. Figure 6 shows that for small $c > 0$, periodic orbits of the expected stability type exist which are close to the periodic orbits for the uncoupled system. (To aid in interpreting this plot, we note that the maximum x values for the stable and unstable periodic orbits are approximately 0.72 and 0.45, respectively.) Indeed, for the uncoupled system the A, B, and 0 solutions are stable; thus, the in phase $A \cdot A$ solutions, out of phase $A \cdot B$ solutions, large-small solutions (which in the limit $c \to 0$ approach the $A \cdot 0$ solutions), and no motion $0 \cdot 0$ solution are all stable for small $|c|$. All solutions which involve an unstable periodic for the uncoupled system as $c \to 0$ are unstable.

We now fix the coupling strength as $c = 0.03$ and take $b = 0.2, F = 0.5$, and treat ω_f as a bifurcation parameter. Figure 7 shows the corresponding bifurcation diagram for the in phase, out of phase, and no motion states. We see that the in

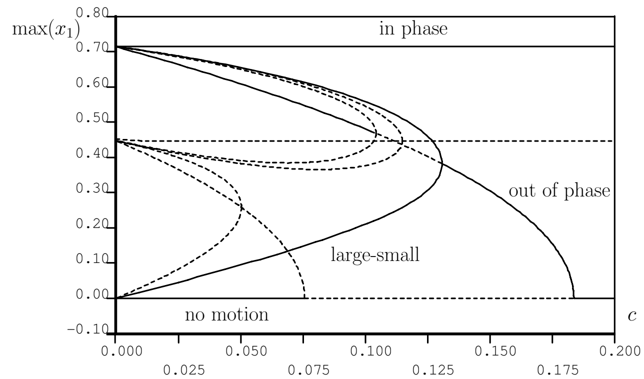

Figure 6. BIFURCATION DIAGRAM FOR FIXED $b = 0.2, F = 0.5, \omega_f = 2.25$ WITH THE COUPLING STRENGTH c TREATED AS A BIFURCATION PARAMETER.

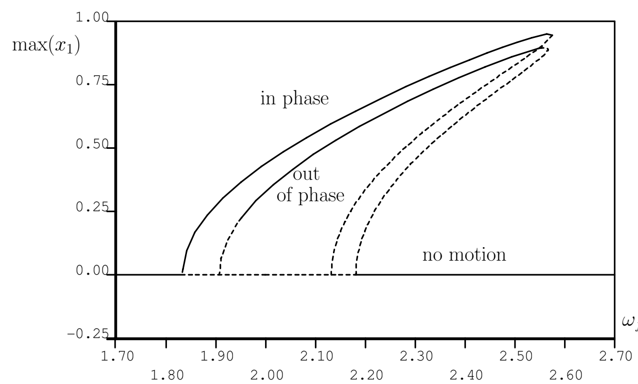

Figure 7. PARTIAL BIFURCATION DIAGRAM FOR EQS. (3,4) FOR FIXED $b = 0.2, F = 0.5, c = 0.03$ WITH ω_f TREATED AS A BIFURCATION PARAMETER.

phase and out of phase states bifurcate from the no motion state at different values of ω_f. As expected from the discussion above, for $\omega_f = 2$ both the in phase and out of phase states are stable, while for $\omega_f = 2.25$ the in phase, out of phase, and no motion states are all stable. We expect at $\omega_f = 2.25$ there will be large-small solutions; this is verified in Fig. 8 which identifies them as being on a branch which bifurcates from the out of phase solution branch. The large-small solution at $\omega_f = 2.25$ is shown in Fig. 9.

Although our analytical results only apply in the weak coupling limit, we note that interesting dynamics occur for larger $|c|$, such as anti-synchronized chaotic behavior for $b = 0.2, F = 0.5, \omega_f = 2$, and $c = -0.9$ shown in Fig. 10 and Fig. 11.

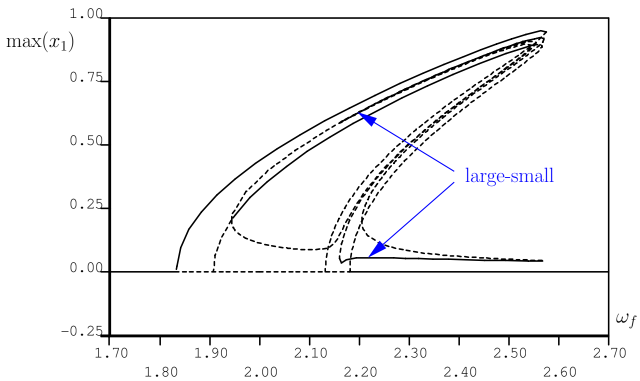

Figure 8. FULL BIFURCATION DIAGRAM FOR EQS. (3,4) FOR FIXED $b = 0.2, F = 0.5, c = 0.03$ WITH ω_f TREATED AS A BIFURCATION PARAMETER.

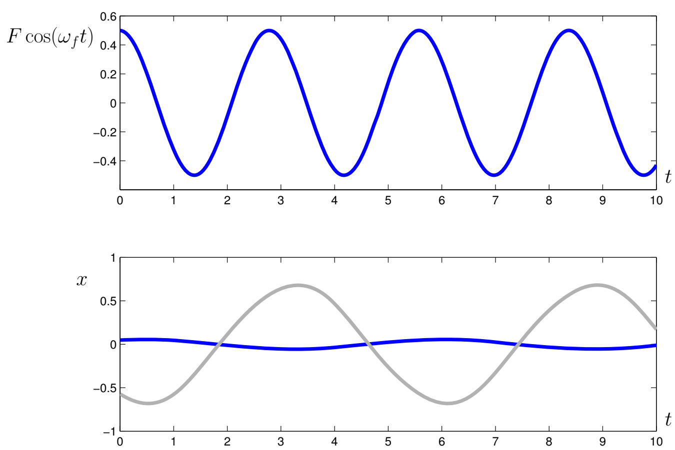

Figure 9. EXAMPLE LARGE-SMALL PERIODIC ORBIT FOR $b = 0.2, F = 0.5, \omega_f = 2.25, c = 0.03$. IN THE LOWER PANEL, THE DARK LINE AND LIGHT LINE CORRESPOND TO THE TWO DIFFERENT OSCILLATORS.

N Coupled Oscillators

These results generalize to N coupled parametrically forced oscillators. Specifically, provided the periodic orbits for the individual oscillators for the uncoupled system are hyperbolic, for every periodic solution which exists for the uncoupled system there will be a nearby periodic orbit for the system with sufficiently small coupling. Furthermore, the periodic orbit will inherit the stability properties from the periodic orbit for the uncoupled system.

For example, consider N parametrically forced oscillators which are coupled linearly to all other oscillators:

$$\ddot{x}_i + b\dot{x}_i + x_i + x_i^3 = x_i F\cos(\omega_f t) + c\sum_{j\neq i}(x_j - x_i), \qquad (18)$$

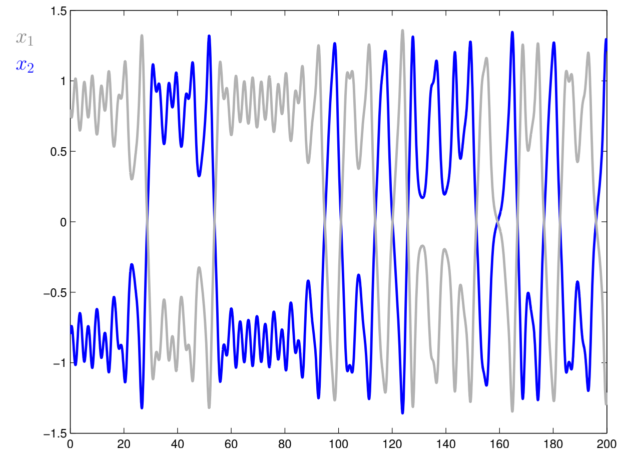

Figure 10. TIME SERIES FOR ANTI-SYNCHRONIZED CHAOS FOR $b = 0.2, F = 0.5, \omega_f = 2$, AND $c = -0.9$, WHERE THE DARK LINE AND LIGHT LINE CORRESPOND TO THE TWO DIFFERENT OSCILLATORS.

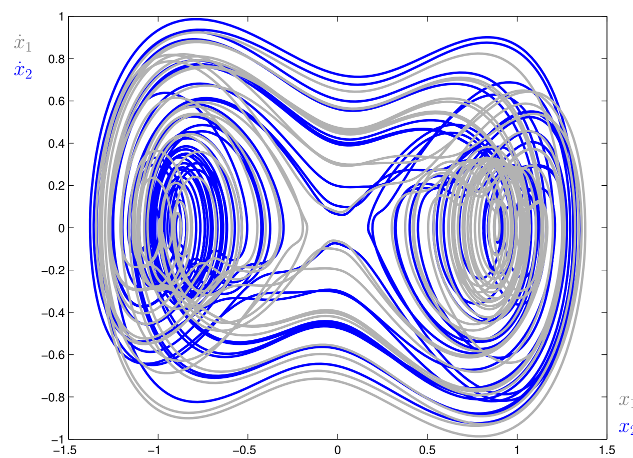

Figure 11. PHASE SPACE REPRESENTATION OF ANTI-SYNCHRONIZED CHAOS FOR $b = 0.2$, $F = 0.5$, $\omega_f = 2$, AND $c = -0.9$, WHERE THE DARK LINE AND LIGHT LINE CORRESPOND TO THE TWO DIFFERENT OSCILLATORS.

where x_i is the position of the i^{th} oscillator, $i = 1, \cdots N$. Suppose $b = 0.2, F = 0.5, \omega_f = 2$, so that when $c = 0$ each oscillator could be in a stable periodic state given by the A and B solutions, or it could be in the unstable no motion state given by the 0 solution. There will be 3^N distinct periodic orbits for $c = 0$: oscillator 1 could be in A, B, or 0, oscillator 2 could be in A, B, or 0, etc. Of these solutions, a total of

$$\frac{N!}{p_A!\,p_B!\,p_0!} \qquad (19)$$

solutions will have p_Y oscillators in each state $Y = A$, B, or 0, where $p_A + p_B + p_0 = N$. This follows from the following combinatorial argument. Suppose we make a list of N symbols such that the i^{th} symbol is A, B, or 0 according to whether the i^{th} oscillator is in the A, B, or 0 state, respectively. In the N slots, there are

$$\binom{N}{p_A} = \frac{N!}{p_A!(N-p_A)!}$$

different ways to put the symbol A in p_A of the slots. Of the remaining $(n - p_A)$ slots, there are

$$\binom{N-p_A}{p_B} = \frac{(N-p_A)!}{p_B!(N-p_A-p_B)!}$$

different ways to put the symbol B in p_B of the slots. The remaining $(N - p_A - p_B)$ slots will have the symbol 0. The product of these is

$$\frac{N!}{p_A!(N-p_A)!} \times \frac{(N-p_A)!}{p_B!(N-p_A-p_B)!} = \frac{N!}{p_A!p_B!p_0!},$$

as in Eq. (19). As an illustration, suppose $N = 4, p_A = 2, p_B = 1$, and $p_0 = 1$. The different possible lists of symbols are

$$AAB0 \quad AA0B \quad ABA0 \quad A0AB \quad AB0A \quad A0BA$$

$$BAA0 \quad 0AAB \quad BA0A \quad 0ABA \quad B0AA \quad 0BAA,$$

giving a total of

$$\frac{4!}{2!1!1!} = 12$$

possibilities.

Interestingly, the implicit function theorem argument presented above does not depend on the coupling topology of the system, or any special properties about the coupling strength (for example, all the strengths being equal). That is, regardless of how the oscillators are coupled together, for sufficiently small coupling strengths there will be analogues of the periodic solutions which exist for the uncoupled system. For example, instead of the oscillators having all-to-all coupling, similar results hold for oscillators coupled only to their neighbors. Of course, as the coupling strengths increase away from 0, so that the above arguments no longer hold, the coupling topology will affect the types of states that exist and are stable.

CONCLUSION

We have discussed periodic solutions which occur for parametrically forced oscillators that are weakly coupled together. In particular, the existence and stability of periodic orbits for the coupled system can be determined by the existence and stability of the individual parametrically forced oscillators when they are uncoupled. Our results follow from an application of the implicit function theorem to an appropriate Poincaré map. The results were confirmed using numerical bifurcation analysis for a specific system. These results may be viewed as an analogue of general results on the existence of phase-locked solutions for weakly coupled autonomous oscillators, as in [14, 17].

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Appendix: The Implicit Function Theorem

We state the implicit function theorem, adapted from [32].

Let $\Phi : \mathbb{R}^N \to \mathbb{R}^M$, $1 \le M < N$, be a continuously differentiable function, written as

$$\Phi(\mathbf{x}) = \Phi(x_1, \cdots, x_N) = (\phi_1(x_1, \cdots, x_N), \cdots \phi_M(x_1, \cdots, x_N)).$$

Suppose

$$\Phi(\mathbf{x}^0) = \Phi(x_1^0, \cdots, x_N^0) = 0,$$

and

$$\det \begin{pmatrix} \left.\frac{\partial \phi_1}{\partial x_{N-M+1}}\right|_{\mathbf{x}^0} & \left.\frac{\partial \phi_1}{\partial x_{N-M+2}}\right|_{\mathbf{x}^0} & \cdots & \left.\frac{\partial \phi_1}{\partial x_N}\right|_{\mathbf{x}^0} \\ \left.\frac{\partial \phi_2}{\partial x_{N-M+1}}\right|_{\mathbf{x}^0} & \left.\frac{\partial \phi_2}{\partial x_{N-M+2}}\right|_{\mathbf{x}^0} & \cdots & \left.\frac{\partial \phi_2}{\partial x_N}\right|_{\mathbf{x}^0} \\ \vdots & \vdots & & \vdots \\ \left.\frac{\partial \phi_M}{\partial x_{N-M+1}}\right|_{\mathbf{x}^0} & \left.\frac{\partial \phi_M}{\partial x_{N-M+2}}\right|_{\mathbf{x}^0} & \cdots & \left.\frac{\partial \phi_M}{\partial x_N}\right|_{\mathbf{x}^0} \end{pmatrix} \neq 0.$$

Then there exists a unique continuously differentiable function $f = (f_1, \cdots, f_M)$ from a neighborhood of $(x_1^0, \cdots, x_{N-M}^0) \in \mathbb{R}^{N-M}$ to a neighborhood of $(x_{N-M+1}^0, \cdots, X_N^0) \in \mathbb{R}^M$ such that

$$\Phi(x_1, \cdots, x_{N-M}, f_1(x_1, \cdots, x_{N-M}), \cdots, f_M(x_1, \cdots, x_{N-M})) = 0.$$