Dynamical Systems with Symmetry - ME225DS
Winter 2008

Homework #6 - Due Thursday, March 20, Jeff's mailbox

1. (50 pts total) The normal form equations for the Hopf bifurcation with O(2) symmetry, truncated at cubic order, are

$$\dot{a}_1 = (\lambda + i\omega)a_1 + A(|a_1|^2 + |a_2|^2)a_1 + B|a_1|^2 a_1, \quad (1)$$
$$\dot{a}_2 = (\lambda + i\omega)a_2 + A(|a_1|^2 + |a_2|^2)a_2 + B|a_2|^2 a_2. \quad (2)$$

Here λ and ω are real, and $A = A_R + iA_I$ and $B = B_R + iB_I$ are complex.

(a) (10 pts) Verify that these equations are equivariant with respect to the actions

$$T_\phi : (a_1, a_2) \to (e^{-i\phi}a_1, e^{i\phi}a_2) \qquad \phi \in [0, 2\pi),$$
$$\kappa : (a_1, a_2) \to (a_2, a_1),$$
$$N_\sigma : (a_1, a_2) \to (e^{i\sigma}a_1, e^{i\sigma}a_2) \qquad \sigma \in [0, 2\pi).$$

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(b) (10 pts) Let

$$a_1 = r_1 e^{i\theta_1}, \qquad a_2 = r_2 e^{i\theta_2}.$$

Rewrite (1,2) as differential equations for $r_1, r_2, \theta_1, \theta_2$.

(c) (15 pts) You'll notice that the $(\dot{r}_1, \dot{r}_2)$ equations from (b) have decoupled from the $\dot{\theta}$ equations, so we can study them as a two-dimensional dynamical subsystem. Find all fixed points for this subsystem, identify the conditions on the parameters that must be satisfied for them to exist, and calculate their stability properties.

(d) (15 pts) Use the above results to verify that the possible bifurcation diagrams for this system are given by Figure 1.

2. (50 points total) The normal form equations for the Hopf bifurcation with D_4 symmetry, truncated at cubic order, are

$$\dot{z}_+ = (\lambda + i\omega)z_+ + A(|z_+|^2 + |z_-|^2)z_+ + B|z_+|^2 z_+ + C\bar{z}_+ z_-^2, \quad (3)$$
$$\dot{z}_- = (\lambda + i\omega)z_- + A(|z_+|^2 + |z_-|^2)z_- + B|z_-|^2 z_- + C\bar{z}_- z_+^2. \quad (4)$$

Here λ and ω are real, and $A = A_R + iA_I$, $B = B_R + iB_I$, and $C = C_R + iC_I$ are complex.

(a) (10 pts) Verify that these equations are equivariant with respect to the actions

$$\kappa_1 : (z_+, z_-) \to (z_+, -z_-), \qquad \kappa_2 : (z_+, z_-) \to (z_-, z_+),$$
$$N_\sigma : (z_+, z_-) \to \mathrm{e}^{i\sigma}(z_+, z_-), \qquad \sigma \in [0, 2\pi).$$

(b) (10 pts) Let

$$z_+ = r^{1/2}\cos(\theta/2)e^{i(\phi+\psi)/2}, \qquad z_- = r^{1/2}\sin(\theta/2)e^{i(-\phi+\psi)/2},$$

and define a new time τ according to $d\tau/dt = r$. Rewrite (3,4) as differential equations for r, θ, ϕ, and ψ.

(c) (15 pts) You'll notice that the $(\dot\theta, \dot\phi)$ equations from (b) have decoupled from the other equations, so we can study them as a two-dimensional dynamical subsystem. Find all fixed points for this subsystem. Relate these fixed points to the periodic solutions for (3,4) guaranteed to bifurcate from the trivial solution from the equivariant Hopf bifurcation theorem. Under what conditions on the parameters B and C do periodic solutions *not* guaranteed by the equivariant Hopf bifurcation theorem exist?

(d) (15 pt) Numerically solve the $(\dot\theta, \dot\phi)$ equations for $B = -2.8 - 3i$, $C = -1 + 2i$. What is the attractor in this subsystem? What does this correspond to in the full (3,4) equations?

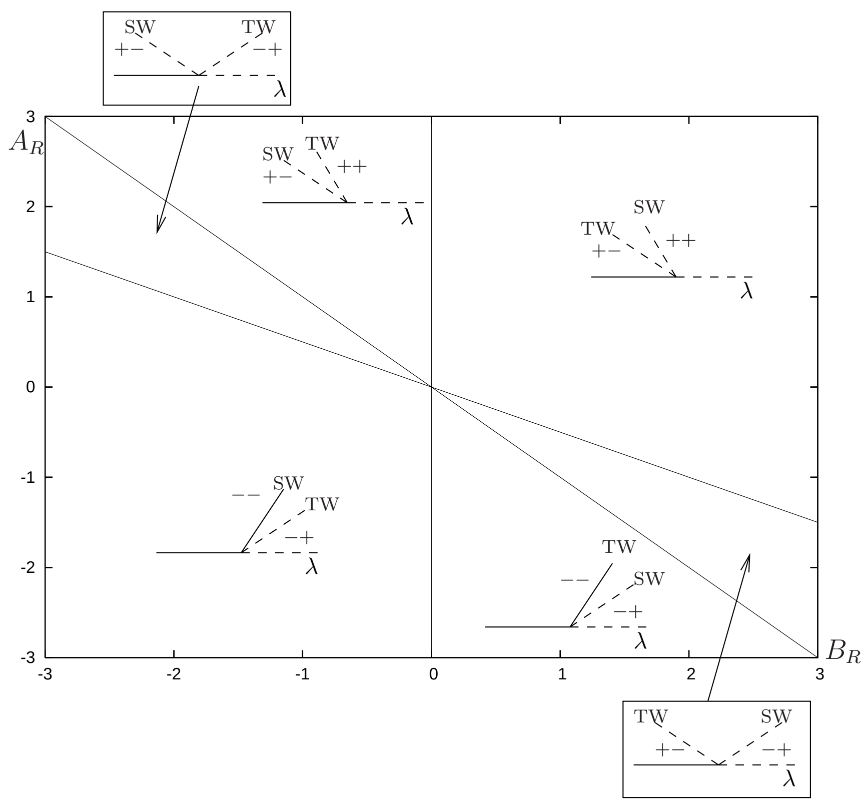

Figure 1: Possible bifurcation diagrams for the normal form equations for the Hopf bifurcation with O(2) symmetry, truncated at cubic order.