Mathematics of Engineering - ME17
Final Exam Review, Spring 2008

Note: boxed formulas are so important that you should have them memorized, and should have a good idea how they can be derived or verified

- Matlab
  - assigning a matrix; view element, row, column of matrix in Matlab
  - `+ - * .* / ./ ^ .^ '`
  - `plot, semilogx, semilogy, loglog, subplot, title, xlabel, ylabel`
  - trig functions, `abs, sqrt, exp, log, log10`
  - `max, min, sum, cumsum, prod, cumprod`
  - logical statements in Matlab: `if` statement, `~ & | ==`
  - functions: external, inline, anonymous
  - loops: `for, while`
  - you will need to write a short Matlab program on the exam!
- probability
  - basic definitions, rules for mutually exclusive events and independent events
  - probability distribution functions
  - expected values
- matrices
  - matrix addition, subtraction, multiplication
  - calculate determinant, trace, eigenvalues/eigenvectors, transpose of matrix

    $$\boxed{\text{determinant} = \text{product of eigenvalues}, \qquad \text{trace} = \text{sum of eigenvalues}}$$

  - inverse of matrix, including "trick" for 2×2 matrices:

    $$\boxed{\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}}$$

  - calculate norms of matrix - don't need to memorize formulas, but should know how to use
  - LU decomposition
  - proofs using indices, for example

    $$(AB)C = A(BC) \qquad \text{trace}(AB) = \text{trace}(BA), \qquad (AB)^T = B^T A^T$$

- basics of complex numbers
  - Euler formula $\boxed{e^{ix} = \cos(x) + i\sin(x)}$
  - going from Cartesian to polar coordinates and vice versa
- basics of calculus
  - Derivatives
    * definition
    * product rule, chain rule, quotient rule
  - Integrals
    * definition as area under a function
    * fundamental theorems of calculus
    * solving using substitution, integration by parts
  - Taylor series

$$\boxed{f(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdots}$$

  - limits: using Taylor series, L'Hospital
  - finding extrema of functions: $f'(x) = 0$, use f'' to determine if min or max
- solving nonlinear equations $f(x) = 0$
  - bisection method, secant method, Newton's method, fixed point iteration: know where these come from and how they can be used
- curve fitting
  - plots using logarithmic axes
    * $y = Ax^b \Rightarrow$ plotting $\log_{10}(y)$ vs. $\log_{10}(x)$ gives straight line
    * $y = A \times 10^{\lambda x} \Rightarrow$ plotting $\log_{10}(y)$ vs. x gives straight line
    * $x = A \times 10^{\lambda y} \Rightarrow$ plotting y vs. $\log_{10}(x)$ gives straight line
- interpolating polynomials, splines, least squares fits
  - don't need to memorize formulas, but should know basics and how to use
- numerical differentiation
  - forward difference

$$\boxed{f'(x) \approx \frac{f(x+h)-f(x)}{h}}$$

  - backward difference

$$\boxed{f'(x) \approx \frac{f(x)-f(x-h)}{h}}$$

  - central difference

$$\boxed{f'(x) \approx \frac{f(x+h)-f(x-h)}{2h}}$$

- numerical integration
  - know how to derive simple rules such as trapezoidal rule geometrically and using systematic/test function method

Some practice problems

1. Suppose $f(x) = \sin(2x)$.

(a) (10 points) Calculate the Taylor expansion of $f(x)$ about the point $x = \pi$ up to and including terms of $\mathcal{O}((x-\pi)^3)$.

(b) (5 pts) Calculate

$$\lim_{x\to\pi} \frac{\sin(2x)}{x-\pi}.$$

(c) (5 pts) What is the maximum value of $f(x)$, and at what x-value does it occur, for $\pi/8 \le x \le \pi/2$?

(d) (5 pts) What is the minimum value of $f(x)$, and at what x-value does it occur, for $\pi/8 \le x \le \pi/2$?

(e) (5 pts) What will the approximation to $f'(\pi/4)$ be when you use the central difference formula? Explain your reasoning.

2. Let

$$A = \begin{pmatrix} 3 & -1 & -6 \\ 2 & 0 & 3 \\ -1 & -6 & 2 \end{pmatrix}.$$

(a) Calculate the matrix norm

$$||A||_1 = \max_{1\le j\le 3} \sum_{i=1}^{3} |A_{ij}|.$$

(b) Calculate the matrix norm

$$||A||_\infty = \max_{1\le i\le 3} \sum_{j=1}^{3} |A_{ij}|.$$

(c) Calculate the Frobenius norm

$$||A||_f = \left(\sum_{i=1}^{3} \sum_{j=1}^{3} A_{ij}^2 \right)^{1/2}$$

(d) Calculate the determinant of A.

3. Suppose that we make the approximation

$$\int_0^{2h} f(x)dx = Af(0) + Bf(2h), \tag{1}$$

and want to determine what values of A and B to take.

(a) Use the test function $f(x) = 1$ in equation (1) to derive an algebraic equation involving A, B, and h.

(b) Use the test function $f(x) = x$ in equation (1) to derive another algebraic equation which must hold.

(c) Solve the algebraic equations in (a) and (b) for A and B, and hence obtain an approximation to the integral.

4. Suppose we know that $x = A \times 10^{\lambda y}$. Suppose also that when we plot y vs. $\log_{10}(x)$ (that is, we plot y on the vertical axis and $\log_{10}(x)$ on the horizontal axis), we get a straight line with slope equal to $-1/3$ and y-intercept equal to $2/3$. What are A and λ equal to?

5. Suppose you have two standard, fair six-sided dice, which you roll one at a time. Treat parts (a)-(d) independently, ie, consider new rolls for each part. Please explain your logic and show intermediate steps, or you won't receive full credit.

(a) What is the probability that the first dice gives an odd number, and the second dice gives an even number?

(b) What is the probability that the second dice shows the same number as the first dice?

(c) What is the probability that the sum of the two dice is equal to 7 or 9?

(d) What is the probability that one of the dice gives an odd number, and one of the dice gives an even number?

6. Using a Taylor expansion for $f(x)$ about a point x_n, derive the prediction for x_{n+1} from Newton's method for finding a solution to $f(x) = 0$.

7. (a) Prove De Moivre's Theorem that

$$[\cos\theta + i \sin\theta]^n = \cos(n\theta) + i\sin(n\theta).$$

(b) For an invertible matrix A, prove that $\det(A^{-1}) = 1/\det(A)$.

8. Suppose that

$$g(x) = \begin{cases} 5x^2 + 2x + 3 \equiv f_1(x) & \text{for } x < 1 \\ ax^2 + 4x + c \equiv f_2(x) & \text{for } x > 1 \end{cases}.$$

Find a and c so that $g(x)$ and $g'(x)$ are continuous at $x = 1$.

9. Suppose that

$$\mathrm{M} = \begin{pmatrix} 4 & 2 \\ 1 & 5 \end{pmatrix}, \qquad \mathrm{N} = \begin{pmatrix} 6 & -2 \\ -1 & 3 \end{pmatrix}.$$

(a) Write down Matlab commands that assign the above matrices to variables named `M` and `N`, respectively. Assume that you have already assigned these matrices in the following.

(b) According to Matlab, what is `M(:,1)`?

(c) According to Matlab, what is `M(1,2)`?

(d) According to Matlab, what is `M*N`?

(e) According to Matlab, what is `M.*N`?

(f) According to Matlab, what is `(M^2-M.^2)'`?

(g) What are the trace, determinant, eigenvalues, eigenvectors, and inverse of M?

10. Calculate the following:

(a)

$$\int e^x \cos x dx$$

(b)

$$\int x e^x dx$$

(c)

$$\int \sin^3 x \cos x dx$$

(d)

$$\frac{d}{dx}\left[\frac{1}{(x^2 + 5x + 1)^3}\right]$$

(e)

$$\frac{d}{dx}(x^2 + 7)^{5/4}$$