Multi-group disease models

Jonathan Dushoff

dushoff@eno.princeton.edu

Beware of mathematics and all who make empty prophecies. The danger already exists that mathematicians have made a covenant with the devil to darken the spirit and confine man to the bonds of Hell.

—St. Augustine

R_0 Revisited

R_0 is the number of people who would be infected by an infectious individual in a susceptible population

In a susceptible population:

$$R_0 = \beta c D.$$

β Probability of transmission

c Contact Rate

D Average Duration of infection

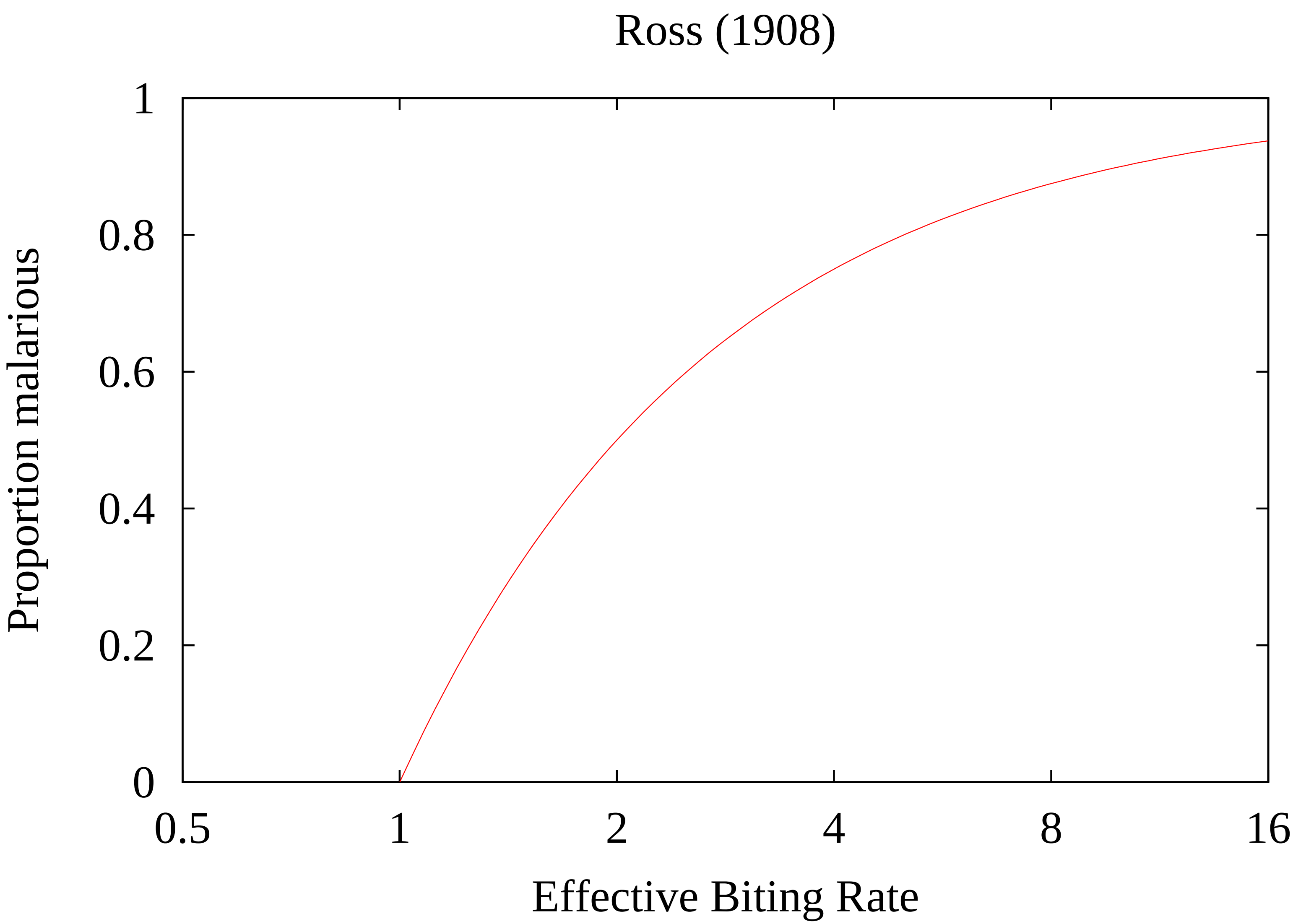
at equilibrium:

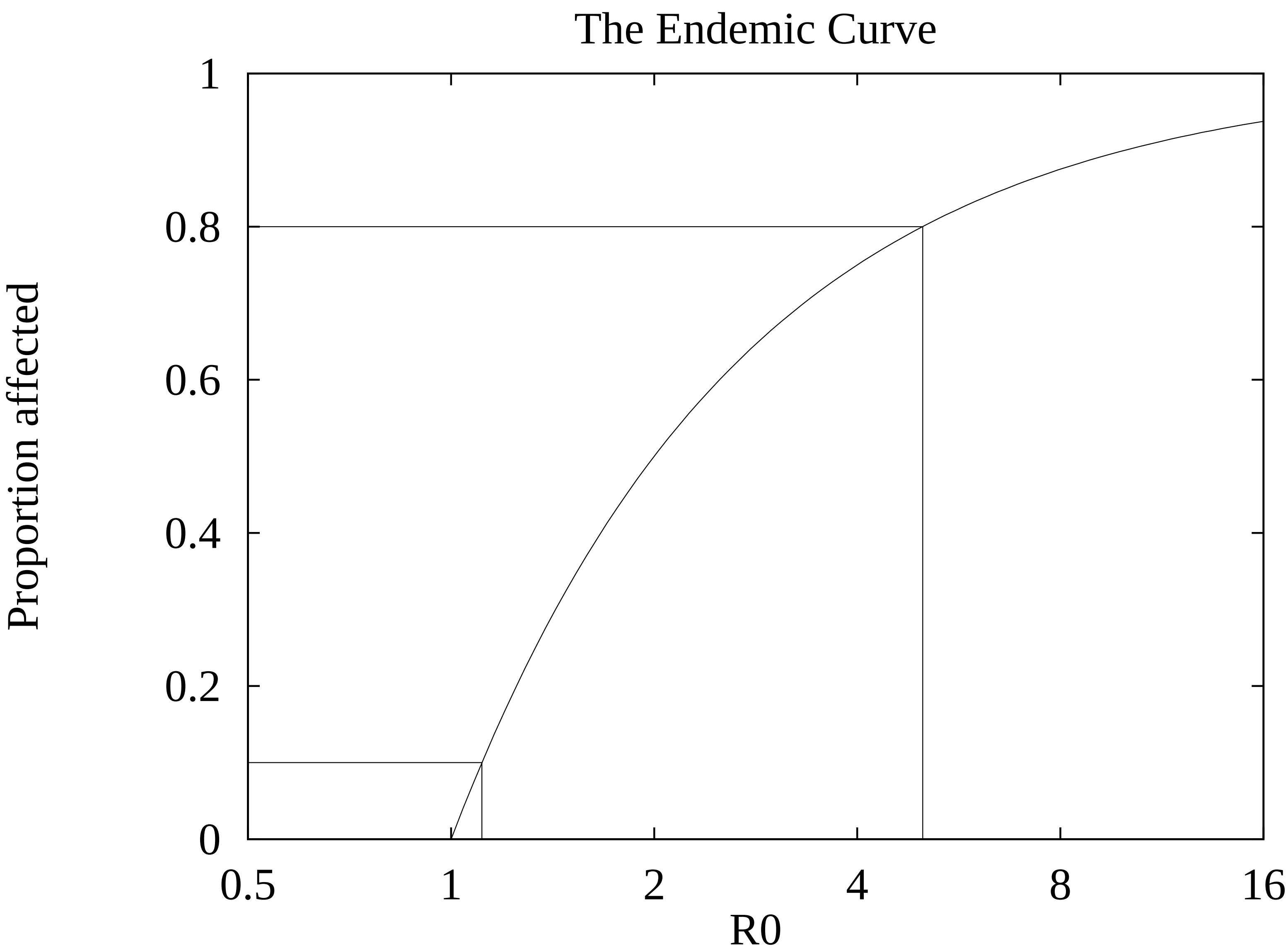
$$1 = \beta c D S.$$

Thus

$$S = 1/R_0.$$

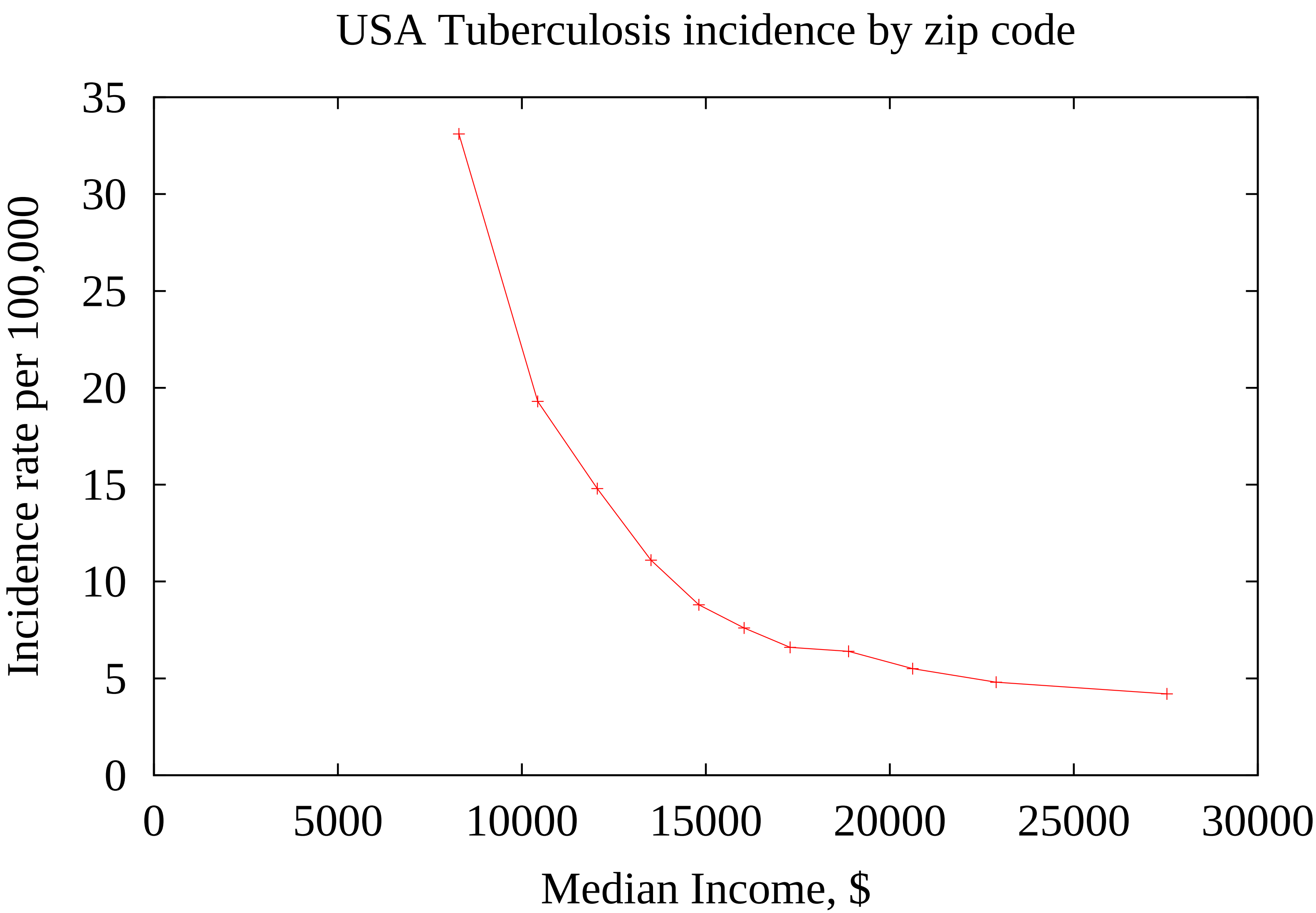
Number ‘affected’ is $V = 1 - S = 1 - 1/R_0$.


Ross (1908)
Proportion malarious
0
0.2
0.4
0.6
0.8
1
0.5
1
2
4
8
16
Effective Biting Rate


The Endemic Curve
Proportion affected
0
0.2
0.4
0.6
0.8
1
0.5
1
2
4
8
16
R0

Host Heterogeneity

- ‘Parametric’ Heterogeneity
  - Susceptibility
  - Transmission
  - Contact Rate
- Spatial Heterogeneity
- Demographic Heterogeneity


USA Tuberculosis incidence by zip code
Incidence rate per 100,000
Median Income, $
0
5
10
15
20
25
30
35
0
5000
10000
15000
20000
25000
30000

A multi-group gonorrhea model

Assume people are differentiated only by mixing rate, and rescale so that the effective mixing rate is equivalent to the ‘subgroup reproductive number’.

$$\dot{I}(a) = a(N(a) - I(a))\Lambda(a) - I(a)$$

where

$$\Lambda(a) = \int p(a,b)\frac{I(b)}{N(b)}db$$

is the proportion of group a’s contacts that are infectious.

The mixing function $p(a,b)$ must satisfy:

$$\int p(a,b)db = 1.$$

and

$$p(a,b)aN(a) = p(b,a)bN(b).$$

A multi-group gonorrhea model

The parameters here are the distribution $N(a)$ and the mixing functions $p(a, b)$.

It will often be more useful to try to estimate the moments of these functions rather than the functions themselves, and to relate R_0 and the proportion affected to these moments (using approximate relationships).

Random mixing

If people mix randomly, then $p(a,b)$ will depend only on subgroup b's importance in the mixing pool:

$$p(a,b) = \frac{bN(b)}{\int cN(c)dc}.$$

The model becomes:

$$\dot{I}(a) = a(N(a) - I(a))\Lambda - I(a)$$

where Λ is now the constant *mixing-weighted* proportion of the population that is infectious:

$$\Lambda = \frac{\int aI(a)da}{\int aN(a)da}$$

Next-generation framework

R_0 is the eigenvalue of the operator that takes the distribution of cases in this generation to the distribution of cases in the next generation.

$$R_0 \hat{I}(a) = aN(a)\hat{\Lambda}(a)$$

We can also use a next-generation approach for the equilibrium:

$$\tilde{I}(a) = a(N(a) - I(a))\tilde{\Lambda}(a)$$

In the random-mixing case, Λ is a constant, so

$$\hat{I}(a) = aN(a),$$

by inspection.

Next-generation framework

Recalling:

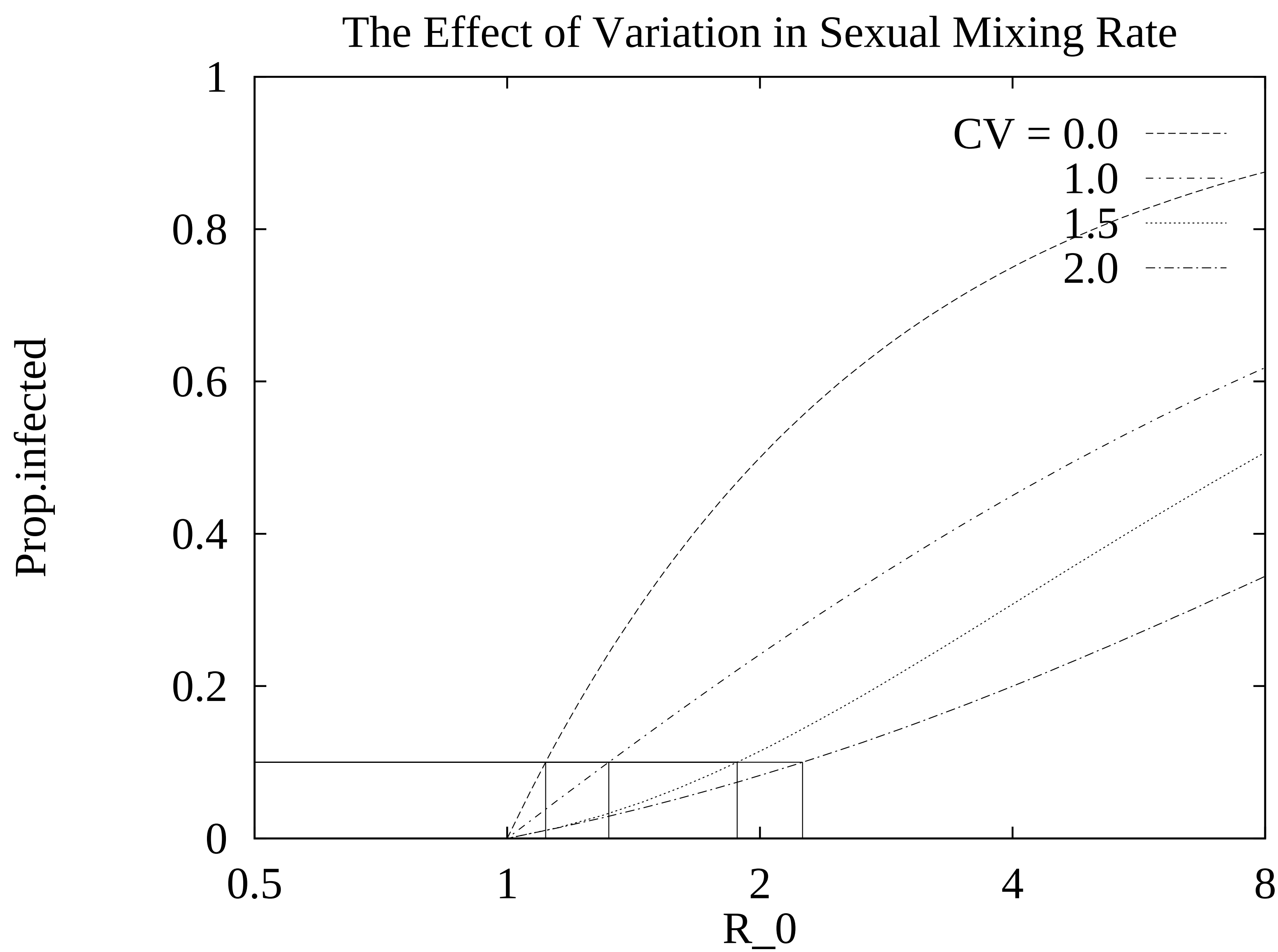
$$\Lambda = \frac{\int aI(a)da}{\int aN(a)da},$$

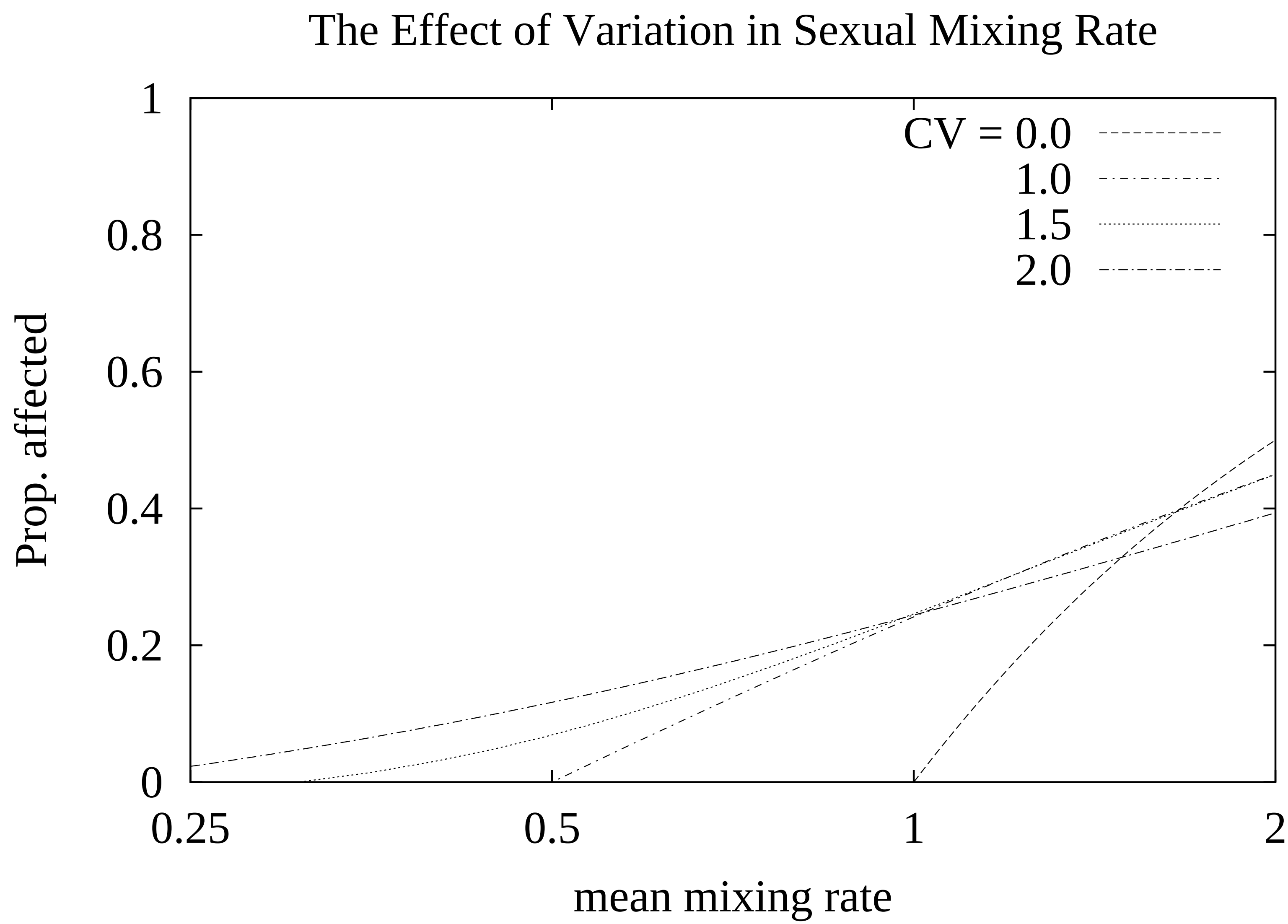
we have

$$R_0 = \frac{\int a^2N(a)da}{\int aN(a)da}.$$

This is the mixing-weighted average of the subgroup reproductive numbers.

We can also write $R_0 = \mu(1 + C^2)$, where μ and $C = \sigma/\mu$ are the mean and CV of the distribution $N(a)$.


The Effect of Variation in Sexual Mixing Rate
CV = 0.0
1.0
1.5
2.0
Prop.infected
1
0.8
0.6
0.4
0.2
0
0.5
1
2
4
8
R_0


The Effect of Variation in Sexual Mixing Rate
Prop. affected
mean mixing rate
CV = 0.0
1.0
1.5
2.0
0
0.2
0.4
0.6
0.8
1
0.25
0.5
1
2

Assortative mixing

Assortative mixing is usually approximated with the somewhat bizarre construct of ‘preferred mixing’, which is not necessarily a bad approximation:

$$p(a,b) = \rho\delta(a,b) + (1-\rho)\frac{bN(b)}{\int cN(c)dc}.$$

Thus

$$\Lambda(a) = \rho\frac{I(a)}{N(a)} + (1-\rho)\Lambda.$$

Assortative mixing

R_0 satisfies

$$\frac{(1-\rho)\displaystyle\int \frac{a^2 N(a)}{R_0 - a\rho}da}{\displaystyle\int aN(a)da},$$

and we can calculate a threshold quantity

$$T_0 = \frac{(1-\rho)\displaystyle\int \frac{a^2 N(a)}{1 - a\rho}da}{\displaystyle\int aN(a)da}.$$

What flattens the endemic curve? (random mixing)

Homogeneous:

$$R_0 = \beta D c$$

$$1 = \beta D c S$$

$$S = 1/R_0$$

Heterogeneous:

$$R_0 = \beta D \hat{c}$$

$$1 = \beta D \tilde{c} S_a$$

$$\tilde{c} < \hat{c}$$

$$S > S_a > 1/R_0$$

What else flattens the endemic curve?

Infected people's sexual partners are more likely than average to be infected, leading to wasted contacts (from the point of view of the disease).

Deterministic Assortative Mixing The tendency of people to mix with people who are in similar neighborhoods or social groups.

Stochastic Demographic Effects The tendency of people to mix with people who are in *the same* neighborhoods or social groups.