

Heteroclinic cycles in coupled cell systems

Michael Field

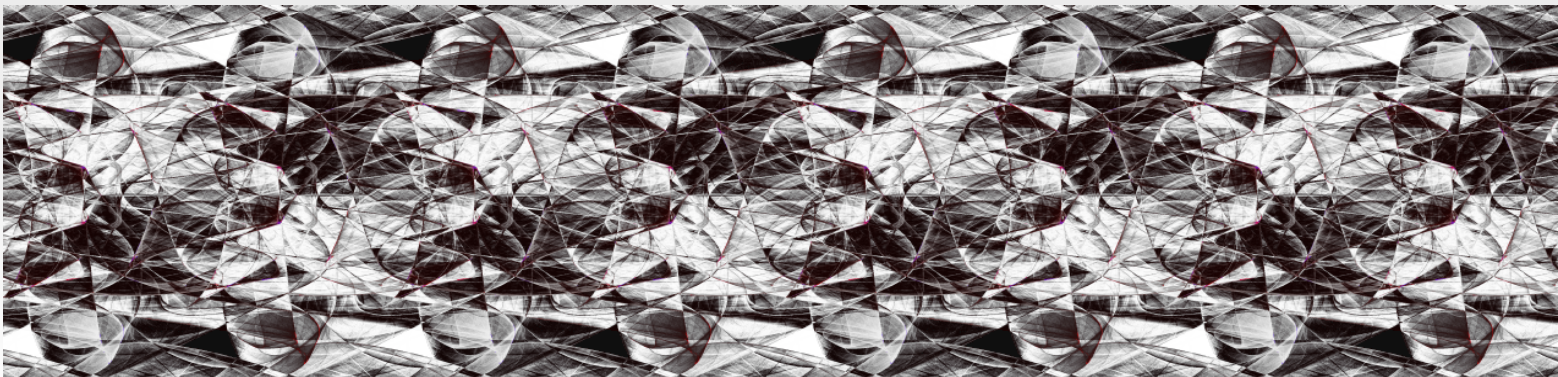
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Some of the research reported on here is joint with Peter Ashwin, Exeter

Extraordinaire comme les mathématiques vous aident à vous connaître

- Samuel Beckett, *Molloy*



Invariant subspaces

Certain classes of dynamical system naturally have *invariant spaces*. That is, subspaces of phase space that are invariant for the dynamics of all systems in the class.

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- ‘Semilinear’ feedback systems.

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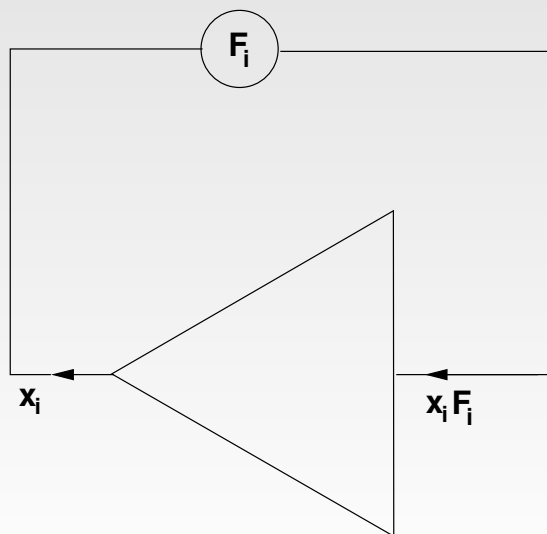
- Symmetric (+ reversible or Hamiltonian) systems.
- Population models based on Lotka-Volterra.
- ‘Semilinear’ feedback systems.
- Coupled cell systems (more later).

Semilinear feedback systems

A semilinear feedback (SLF) system is a set of ODEs

$$\dot{\mathbf{x}}_i = f_i(\mathbf{x}_i) + \mathbf{x}_i F_i(\mathbf{x}_1, \dots, \hat{\mathbf{x}}_i, \dots, \mathbf{x}_n),$$

where $\mathbf{x}_i \in \mathbb{R}^{n_i}$, $1 \leq i \leq n$ and $f_i(0) = 0$. The evolution of \mathbf{x}_i according to $\dot{\mathbf{x}}_i = f_i(\mathbf{x}_i)$ is modified by the linear feedback term F_i which depends typically nonlinearly on the remaining variables.

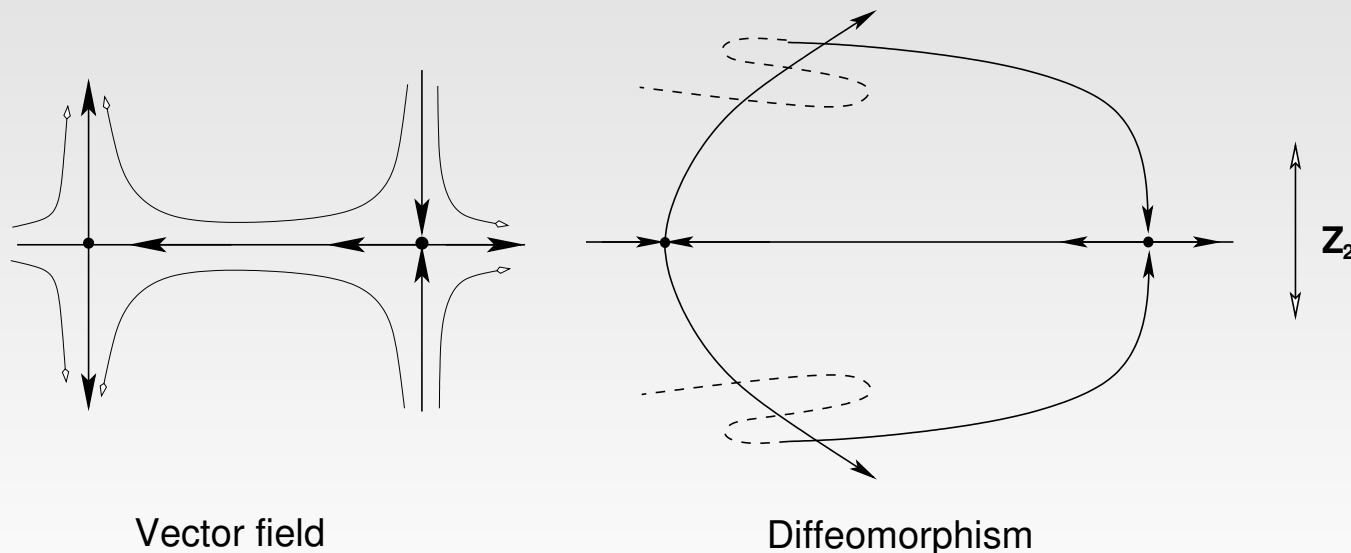


Lotka-Volterra population models are **SLF** systems with F_i linear. Many (not all) examples of symmetric systems are also of this type – at least if we restrict to cubic truncations (F_i will typically be quadratic).

Note that $\mathbf{x}_i = 0$ will always be an invariant subspace for the dynamics of an **SLF** system. Since intersections of invariant subspaces are invariant, $\mathbf{x}_{i_1} = \dots = \mathbf{x}_{i_s} = 0$ is also invariant, $1 \leq i_1 < \dots < i_p \leq n$. If we add symmetry and/or reversibility to the mix, this will result in more invariant subspaces which may or may not play a role in the formation of heteroclinic cycles.

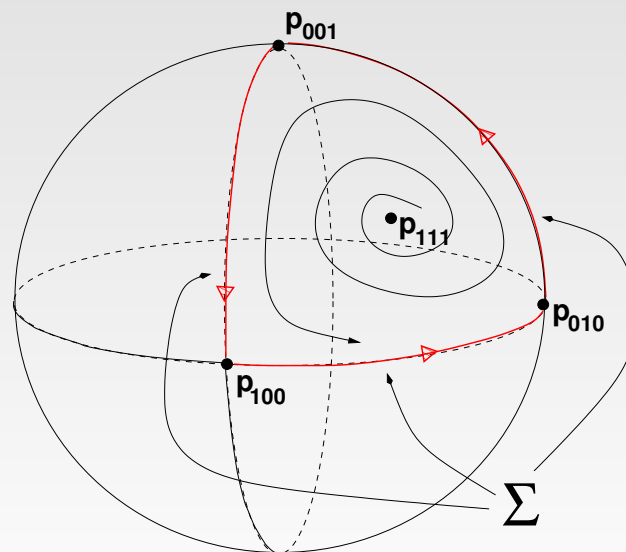
Dynamics & Intersections

The presence of invariant subspaces allows the existence of robust (stable) non-transverse intersections of invariant manifolds of equilibria and limit cycles. In the simplest cases, these will be saddle connections – see figure. Generally, intersections will be singular (all this is understood and covered by the theory of equivariant transversality).



Heteroclinic cycles

In both equivariant dynamics and **SLF** models (in particular, population models), it is possible to have robust cycles of non-transverse saddle connections. First observed by May & Leonard (1975) (population dynamics), later by Dos Reis (1978) (equivariant dynamics on surfaces) and then by Guckenheimer and Holmes (1988) (equivariant bifurcation theory).



Dynamics on flow-invariant attracting sphere.

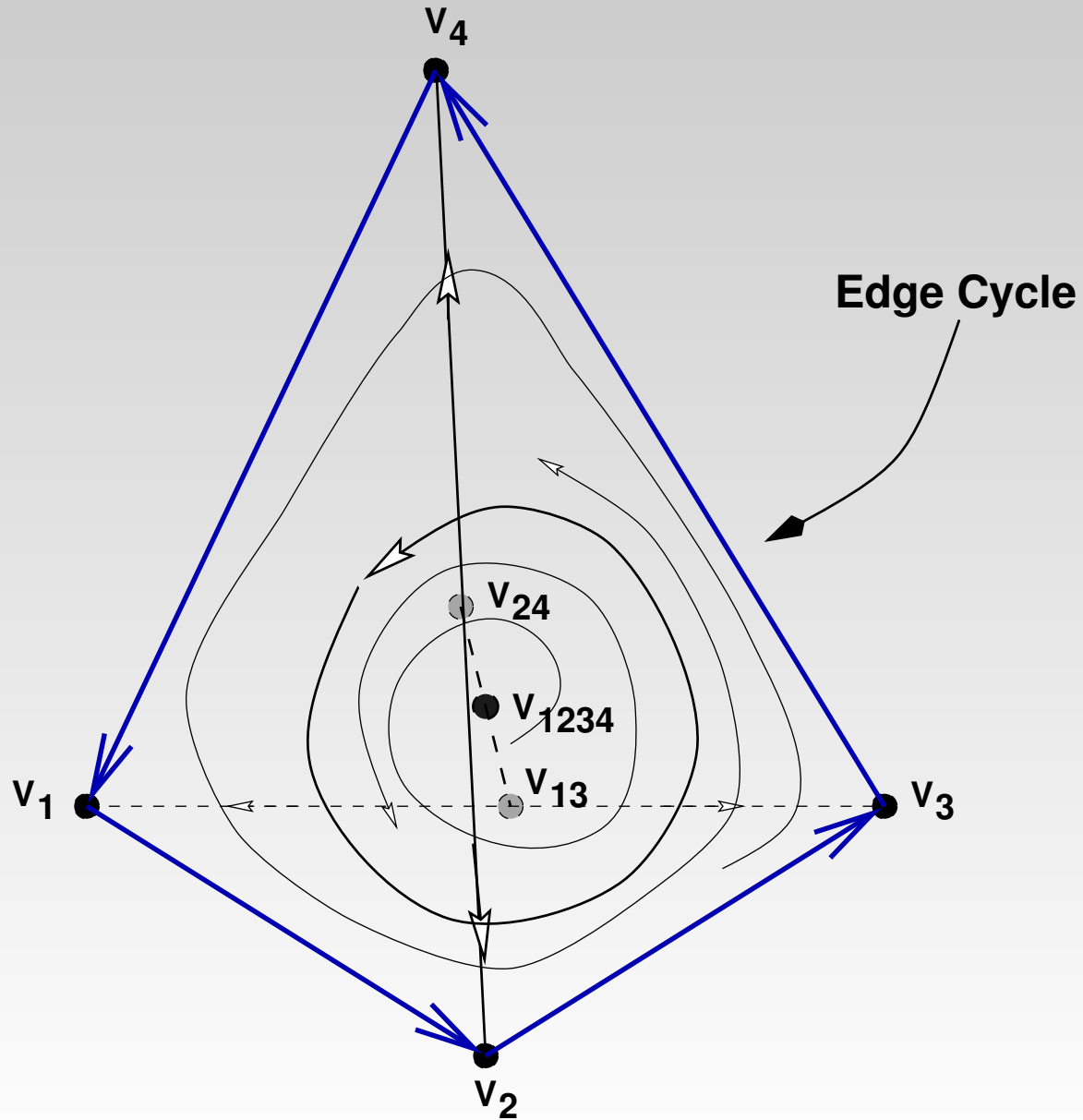
Symmetry group: $\mathbf{Z}_2^3 = \mathbf{Z}_3$

Note the attracting heteroclinic cycle Σ

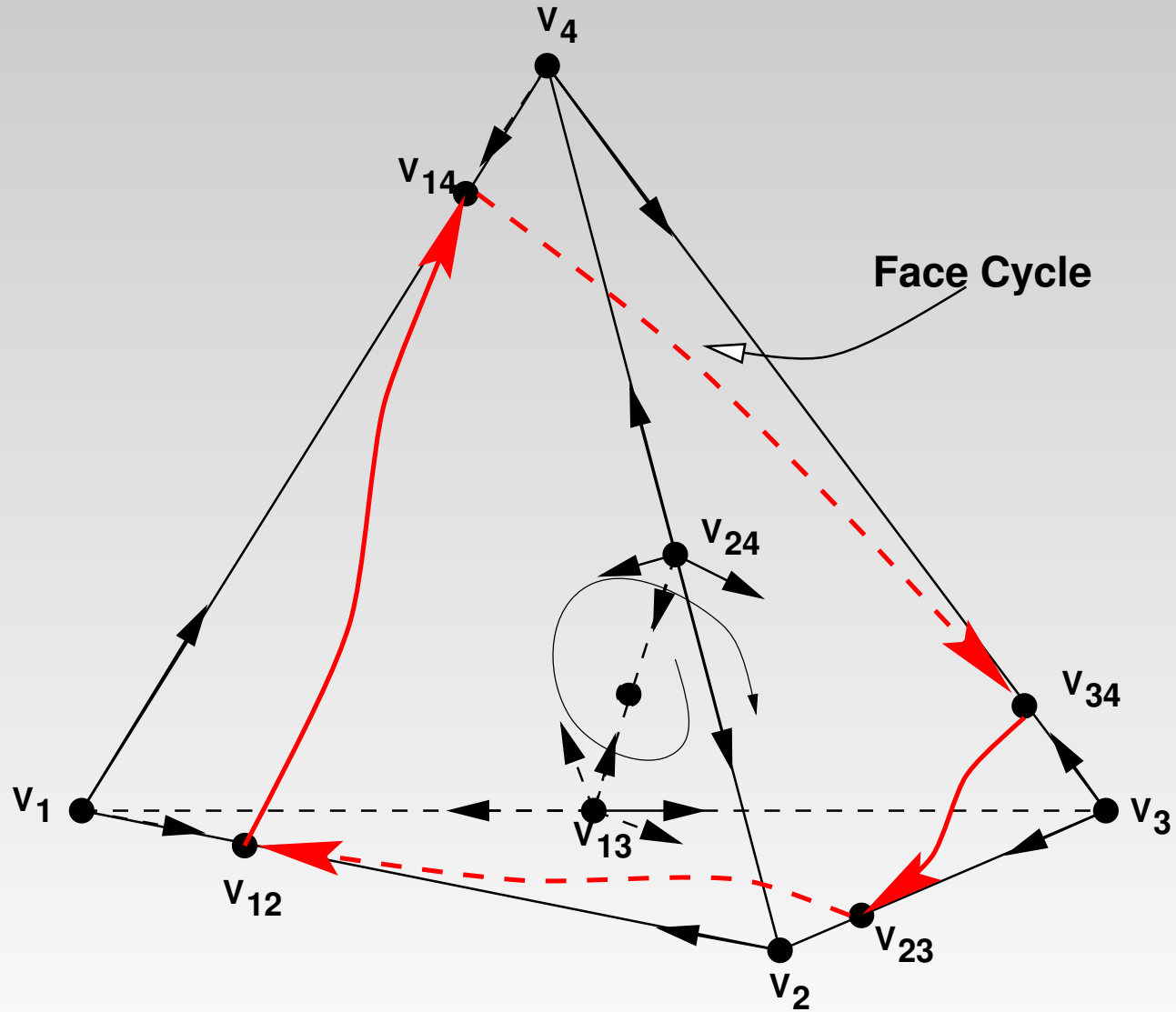
Examples & Models

If we look at dynamics on \mathbb{R}^n equivariant with respect to $\mathbb{Z}_2^n \rtimes \mathbb{Z}_n$ (or just \mathbb{Z}_2^n), there are infinite families of robust attracting heteroclinic cycles. We describe the case $n = 4$. We do this by showing dynamics on the positive orthant of S^3 – invariant under the flow of $\mathbb{Z}_2^4 \rtimes \mathbb{Z}_4$ -equivariant dynamics on S^3 . (A population dynamicist might look at the invariant simplex $x_1 + \dots + x_4 = 1$ in the positive orthant of \mathbb{R}^4). We assume symmetry here but the phenomena we show are characteristic of **SLF** systems and have nothing to do with symmetry. Indeed, from our perspective, all of our examples should be viewed more as phenomenological models.

Edge cycles



Face cycles



Phenomenology

If we take the N -dimensional simplex Δ_N , $N \geq 3$, and choose $1 \leq p \leq N - 2$, we can construct (attracting) heteroclinic cycles connecting equilibria (or periodic orbits or chaotic sets) on the p -dimensional faces of Δ_N . For example, if $N = 5$, $p = 3$, there exist attracting heteroclinic cycles

$$\dots \rightarrow \mathbf{123} \rightarrow \mathbf{234} \rightarrow \mathbf{345} \rightarrow \mathbf{451} \rightarrow \mathbf{512} \rightarrow \mathbf{123} \rightarrow \dots$$

These cycles may connect equilibria, periodic orbits or chaotic sets (“cycling chaos”).

Cycling chaos example: $N = 3, p = 1$

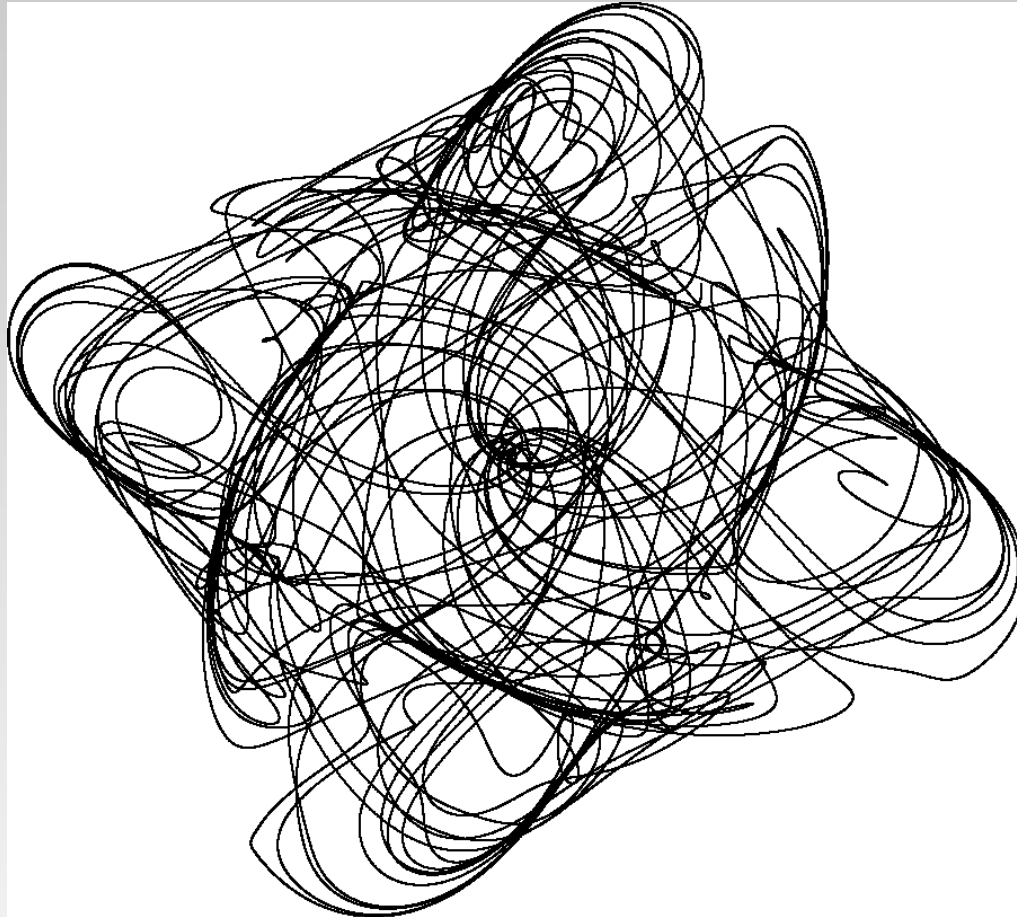
This example is built on the following system of ODEs defined on \mathbb{C}^2 :

$$\begin{aligned}z_1' &= z_1 - a(|z_1|^2 + |z_2|^2)z_1 + \beta \bar{z}_1^2 \bar{z}_2 + \gamma z_2^3, \\z_2' &= z_2 - a(|z_1|^2 + |z_2|^2)z_2 + \beta z_1 \bar{z}_2^2 + \gamma \bar{z}_1^3.\end{aligned}$$

$$a, \beta, \gamma \in \mathbb{R}.$$

Choose $a \gg 0$ so that there is a globally attracting invariant sphere for the dynamics.

Dynamics, projection on y_1, y_2



We show the projection of the phase portrait onto the (y_1, y_2) -plane when $\beta = -1.5, \gamma = 0.29$.

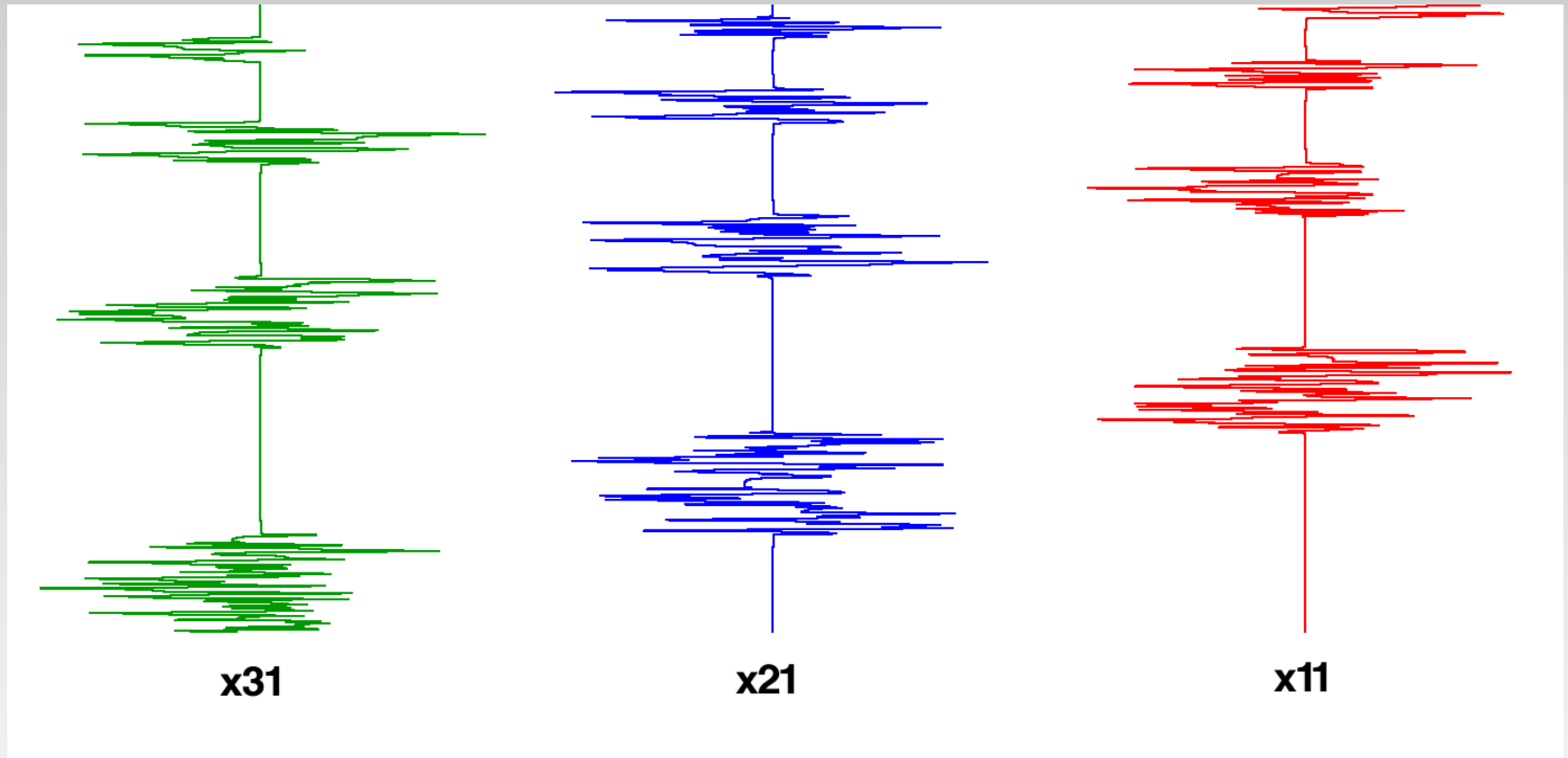
Model system

We couple three of these systems together to obtain the \mathbb{Z}_3 -equivariant system

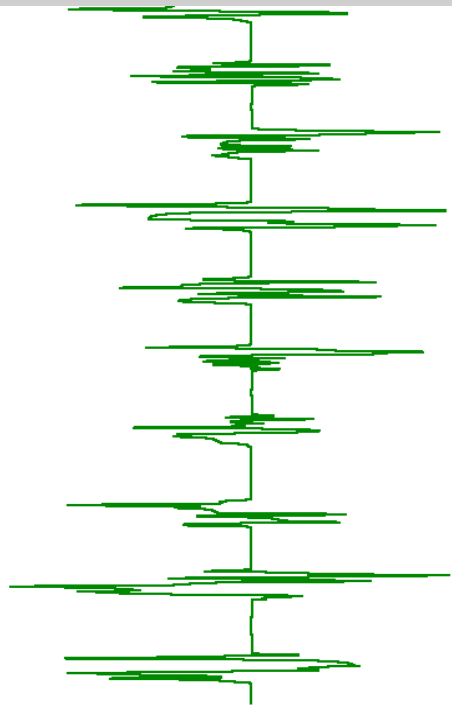
$$\begin{aligned}Z'_1 &= F(Z_1) + \tau \|Z_2\|^2 Z_1, \\Z'_2 &= F(Z_2) + \tau \|Z_3\|^2 Z_2, \\Z'_3 &= F(Z_3) + \tau \|Z_1\|^2 Z_3.\end{aligned}$$

Here τ is a real parameter and we have written the equation of the basic cell in the form $Z' = F(Z)$, where $Z = (z_1, z_2) \in \mathbb{C}^2$. We write the real coordinates for the i th. cell as $(x_{i1}, y_{i1}, x_{i2}, y_{i2})$.

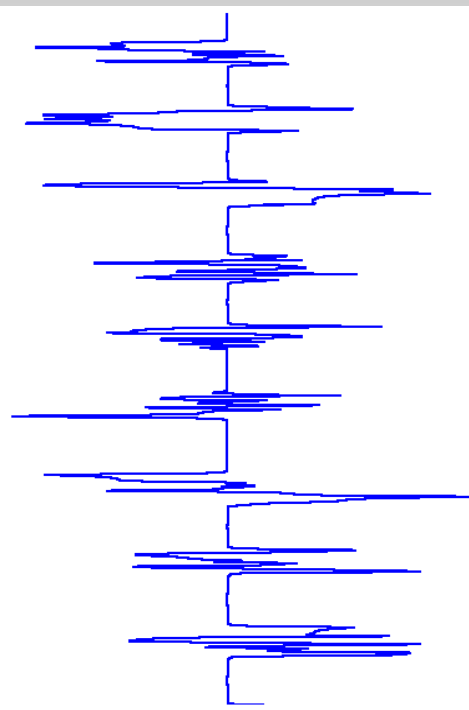
Cycling chaos: $\tau = -2.2$



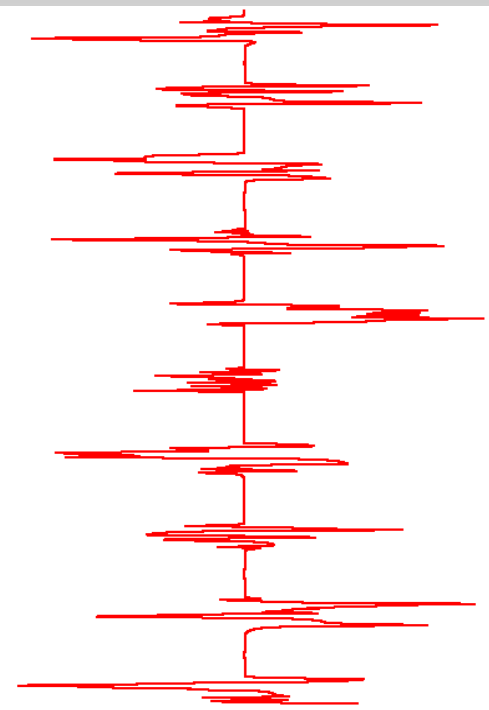
Periodic chaos: $\tau = -1.97$



x31



x21



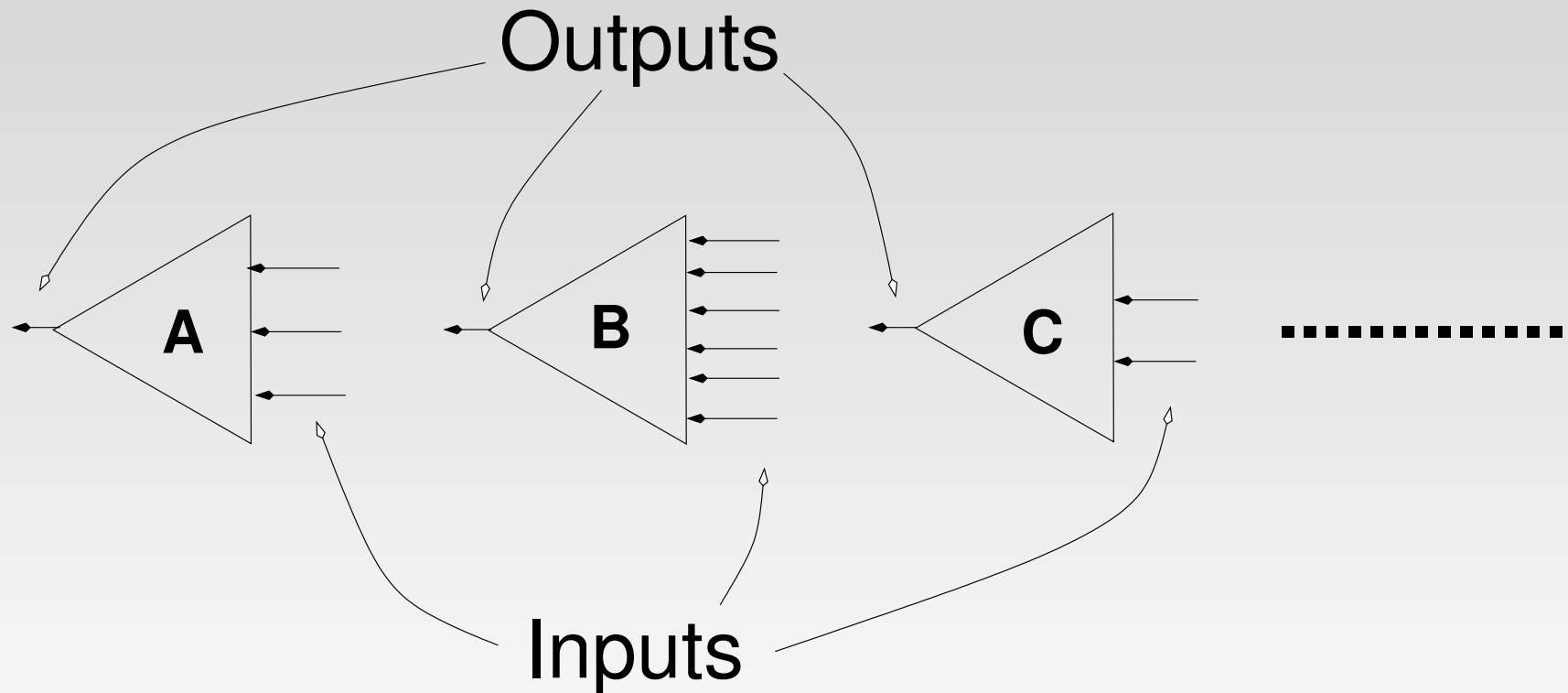
x11

Program

Translation from various types of heteroclinic behaviour that occur in symmetric systems to analogous behaviour in coupled cell systems (no symmetry).

Coupled cell systems: Cell types

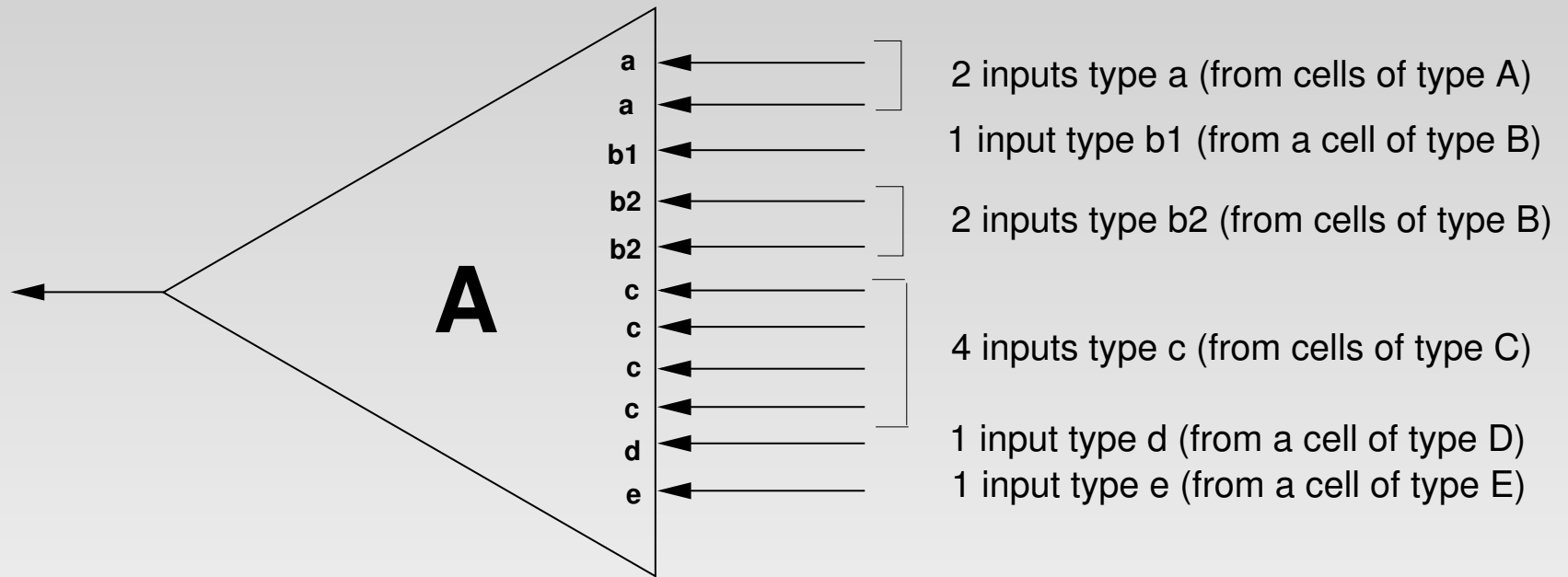
We shall be looking at a finite collection of different cell *types*. We write these **A**, **B**, **C**, Each cell has a finite number of inputs and an output.



Cells: Inputs

Output
(type A)

Inputs



A given cell type may receive inputs from cells of various types. In the figure, a cell of type **A** receives inputs from cells of types **A**, **B**, **C**, **D** and **E**.

Patchcord rules

We interconnect cells using *patchcords*. A type **a** patchcord goes from the output of a cell of type **A** to the **a** input of a cell. If there are type **a1**, **a2**,... inputs, then we color code patchcords so as to indicate which type of input the cord should be patched into.

There are no restrictions on the number of outputs we take from a cell.

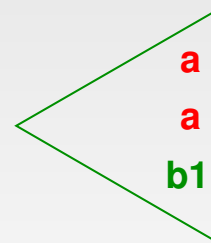
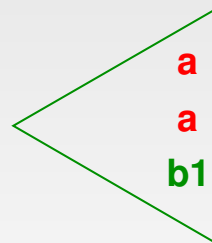
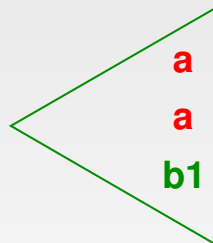
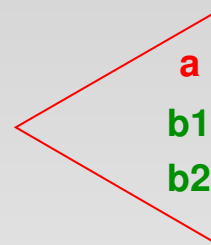
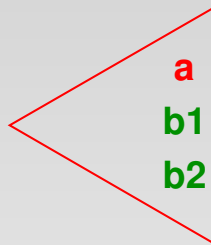
No more than one patchcord is plugged into a given input.

Normally we regard patchcords as ‘dynamically neutral’. However, patchcords could include, for example, a delay line.

Example

Type A: red

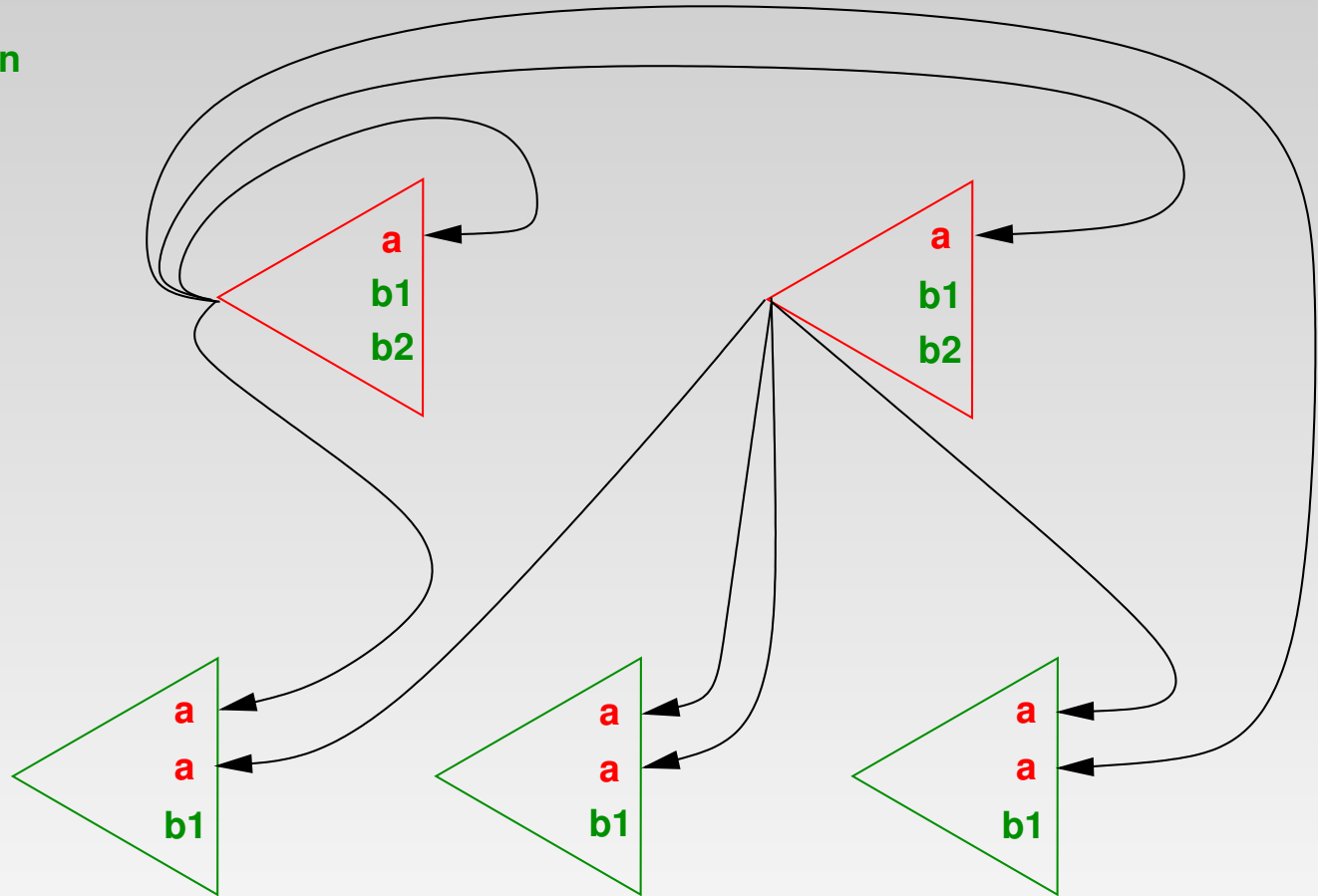
Type B: green



Patching the a inputs.

Type A: red

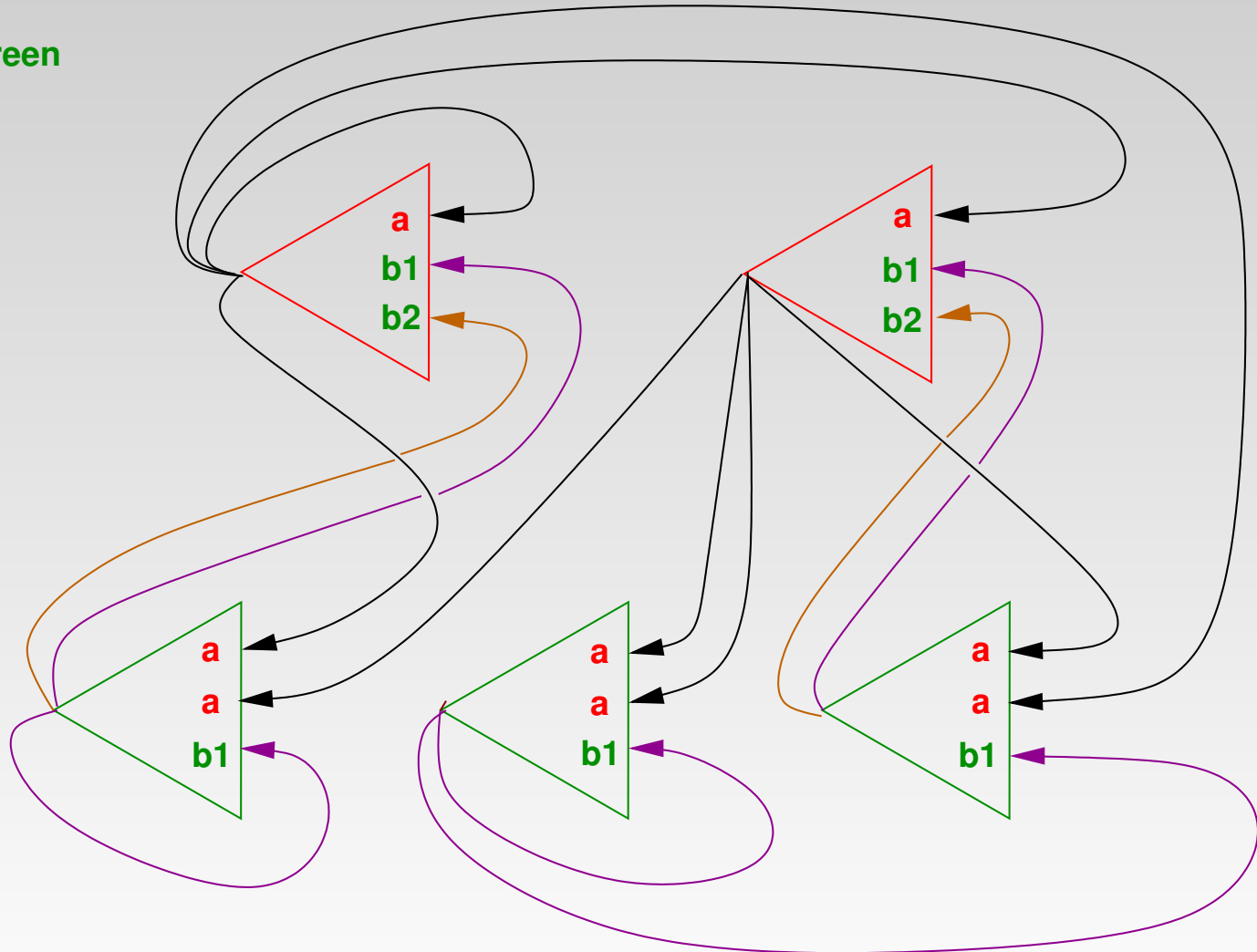
Type B: green



Patching the b inputs.

Type A: red

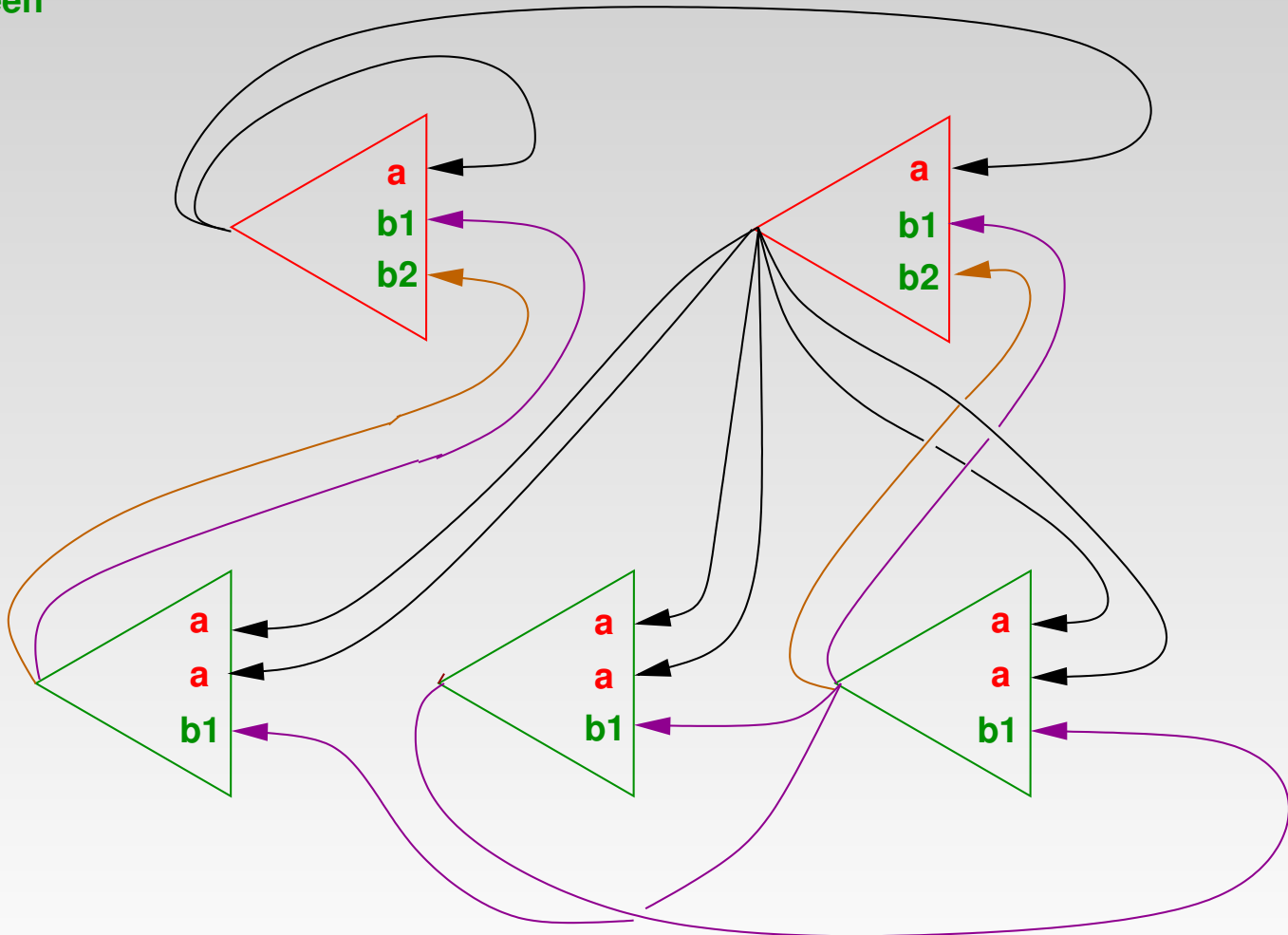
Type B: green



Another Patching.

Type A: red

Type B: green



Coupled cell systems

For us, a coupled cell system will consist of

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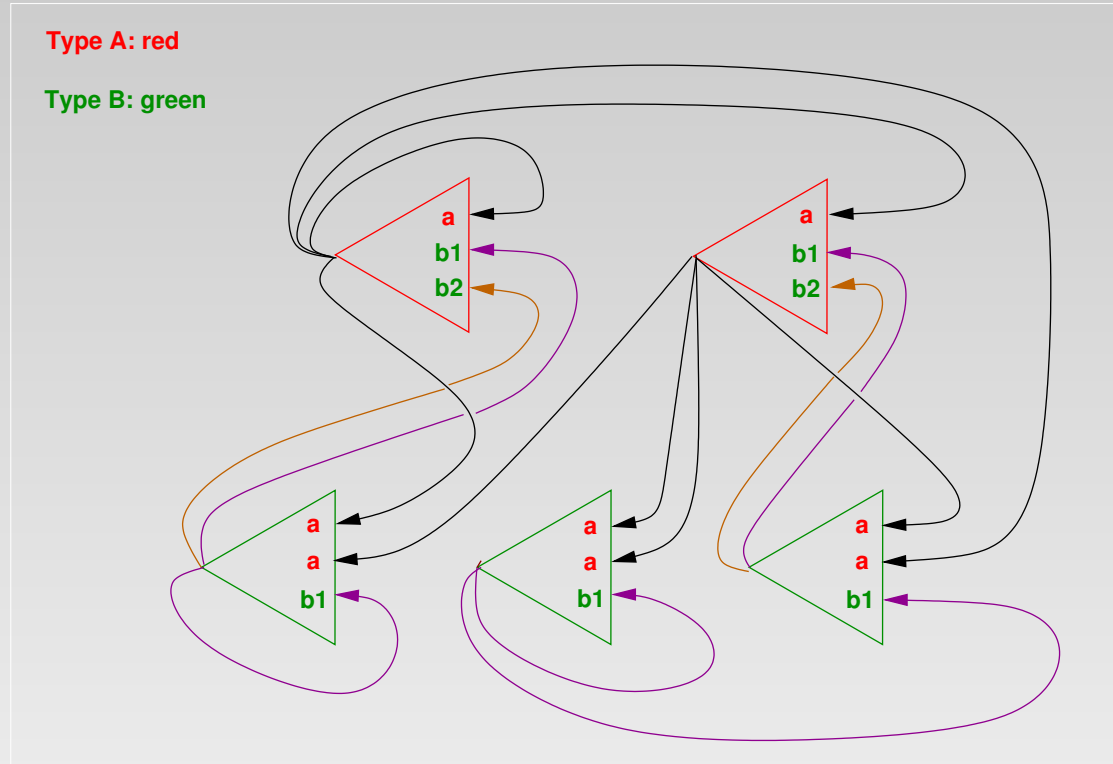
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- There are no restrictions on the number of outputs from a cell of given type.
- Evolution of cells governed by ODEs.

Invariant subspaces

Given: a coupled cell system. We are interested initially in synchronised solutions of the system. These correspond to certain types of invariant subspace of the phase space.

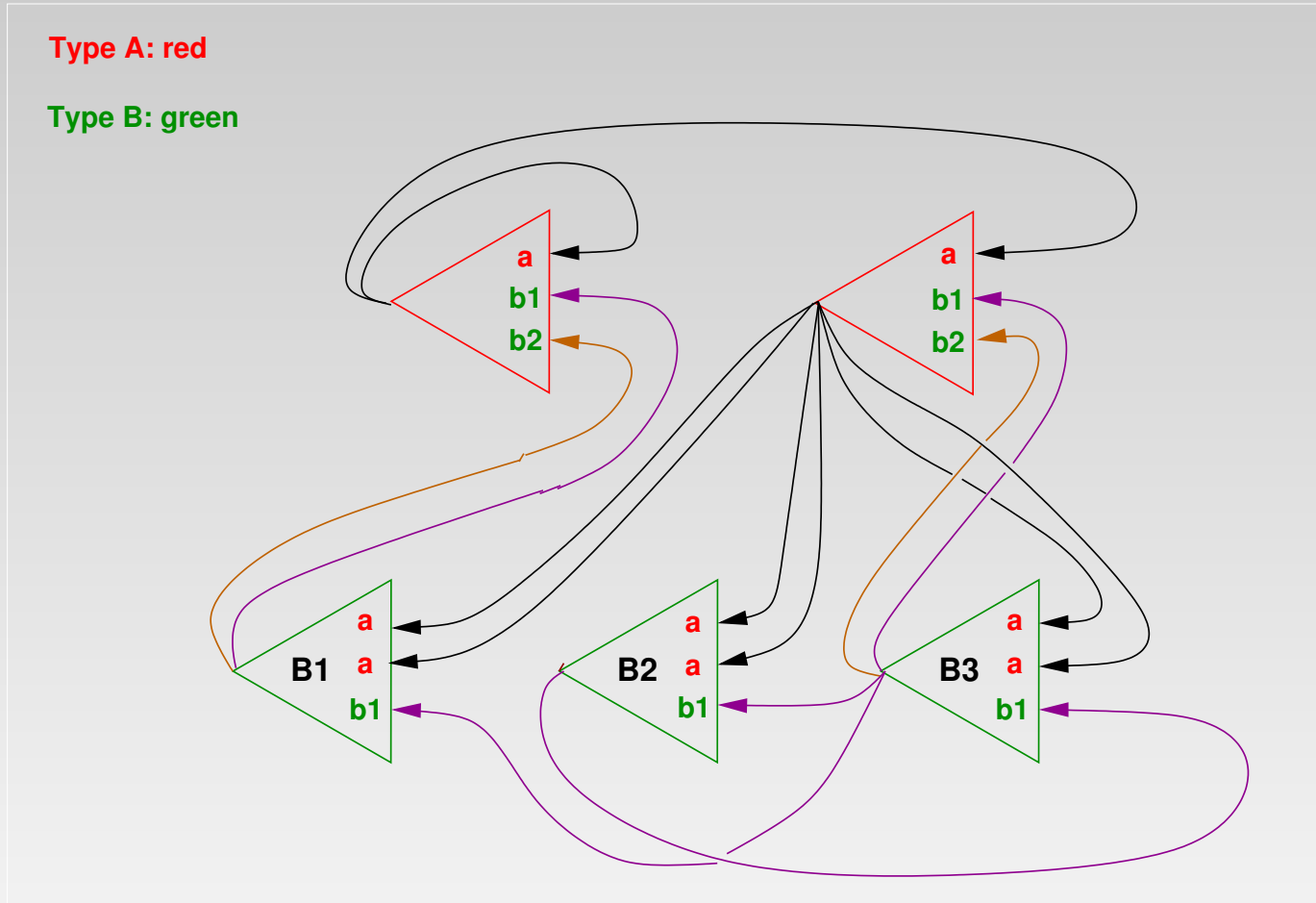
We illustrate the ideas with two very simple examples.

Five cell system I



The only invariant subspace of synchronous solutions corresponds to all the type **A** cells being synchronised *and* all the type **B** cells being synchronised. (This property is true for *all* coupled cell networks – trivial synchronised state.)

Five cell system II



It is possible for just the **B** cells to be synchronised
and for just **B1** (or **B3**) and **B2** to be synchronised.

Repatching rules

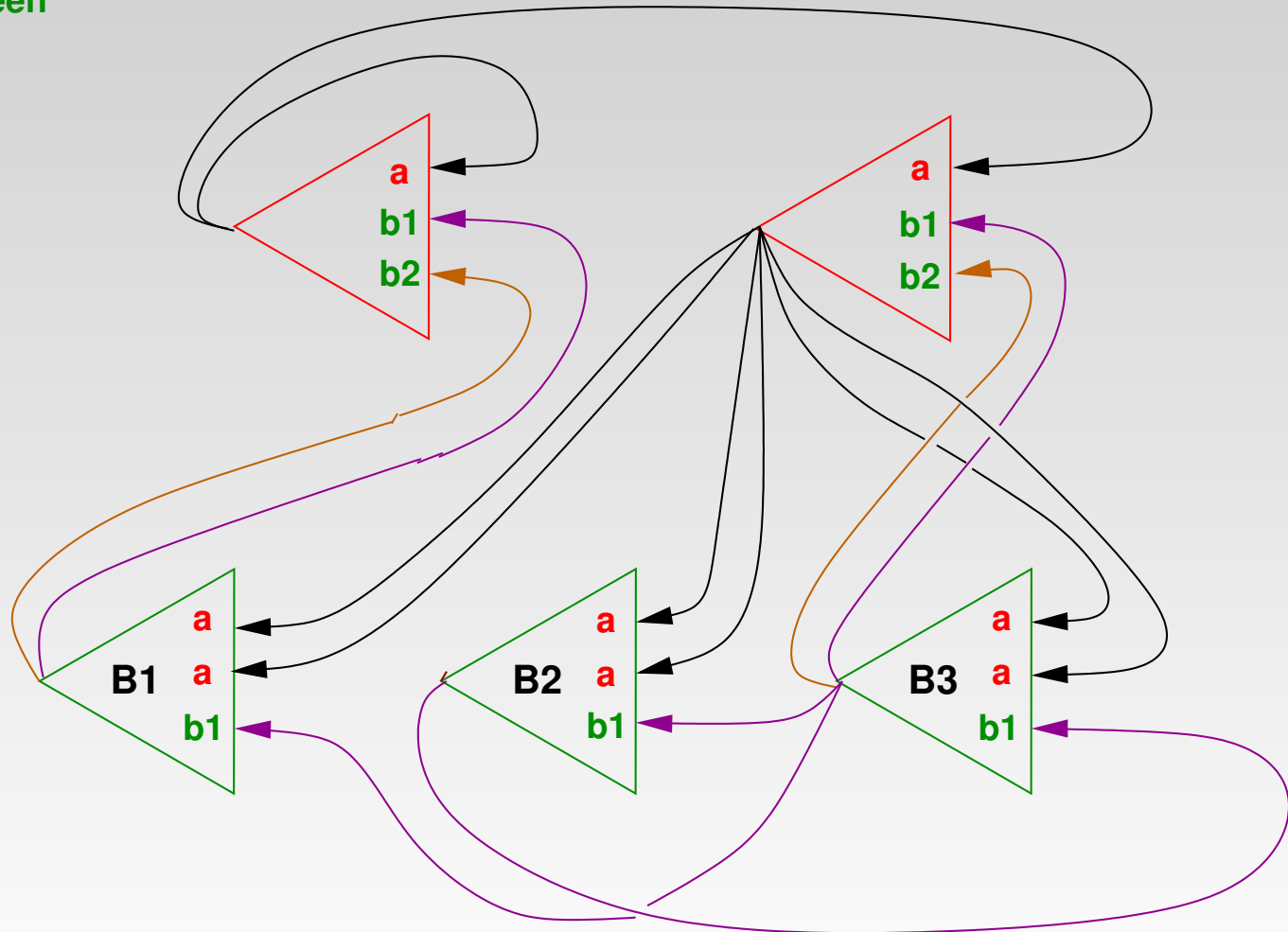
Each class of synchronised solutions determines a *repatching rule* that neither destroys the invariant subspace nor the dynamics on the invariant subspace. The repatching rule defines an equivalence relation on the set of cells or, equivalently, defines a **partition** of the cells (a ‘balanced’ partition). Conversely, a balanced partition (or set of admissible repatching rules) determines a unique invariant subspace of synchronised solutions. Repatching yields a unique minimal network with a number of cells equal to the number of elements of the partition – this is trivial. Dynamics of the minimal network determine dynamics on the associated invariant subspace.

We illustrate this for the previous example.

Repatching 1

Type A: red

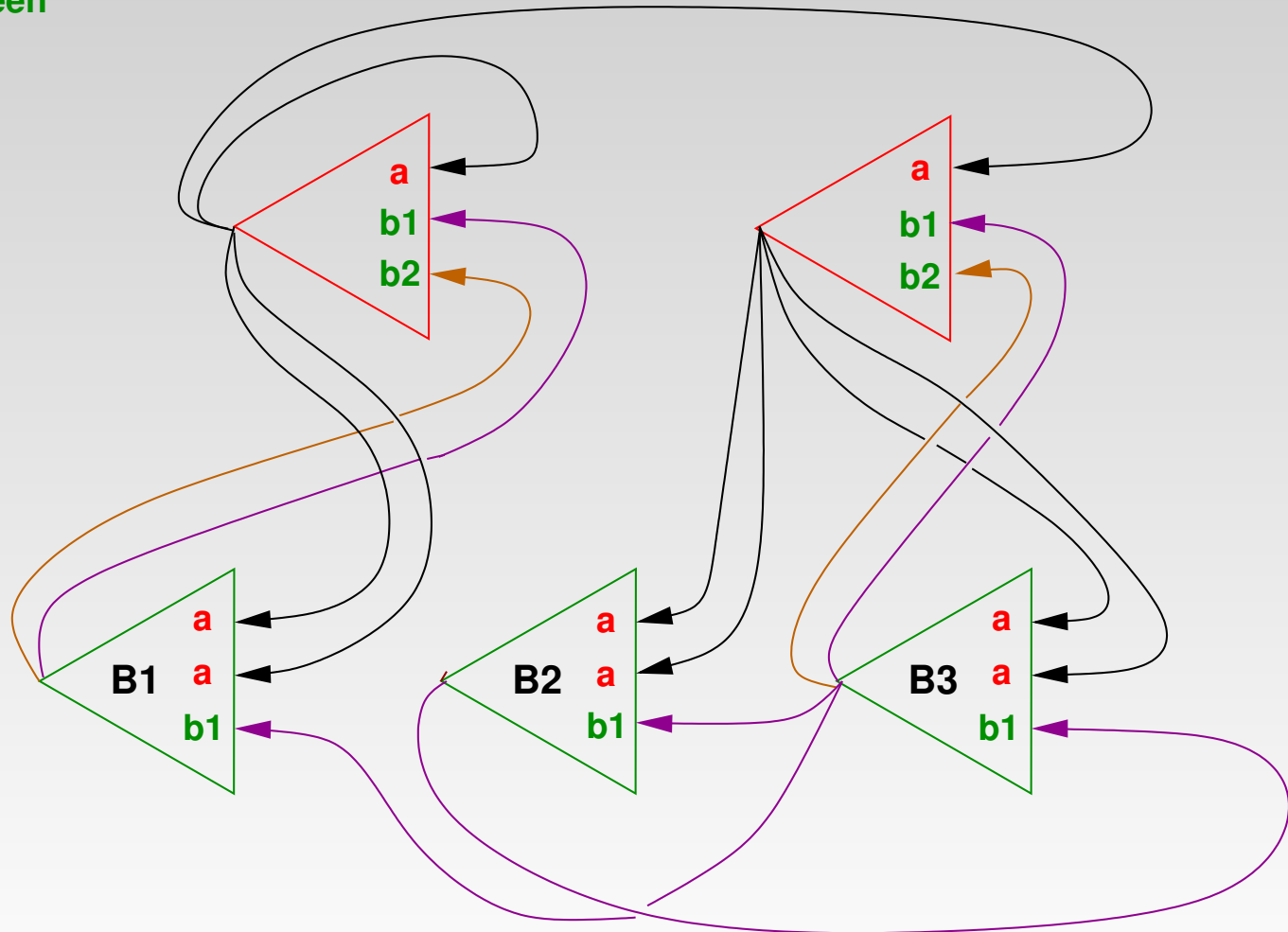
Type B: green



Repatching 2

Type A: red

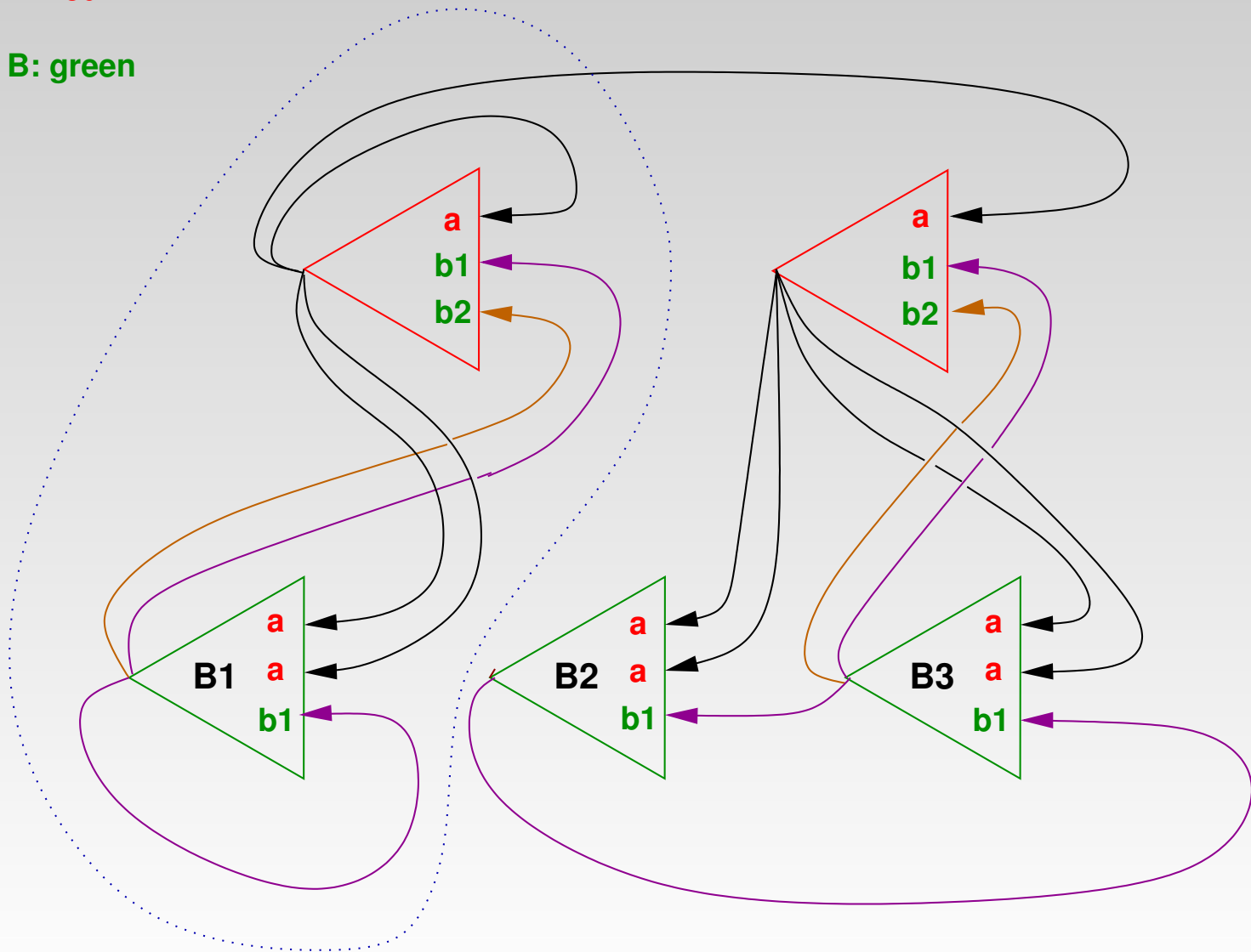
Type B: green



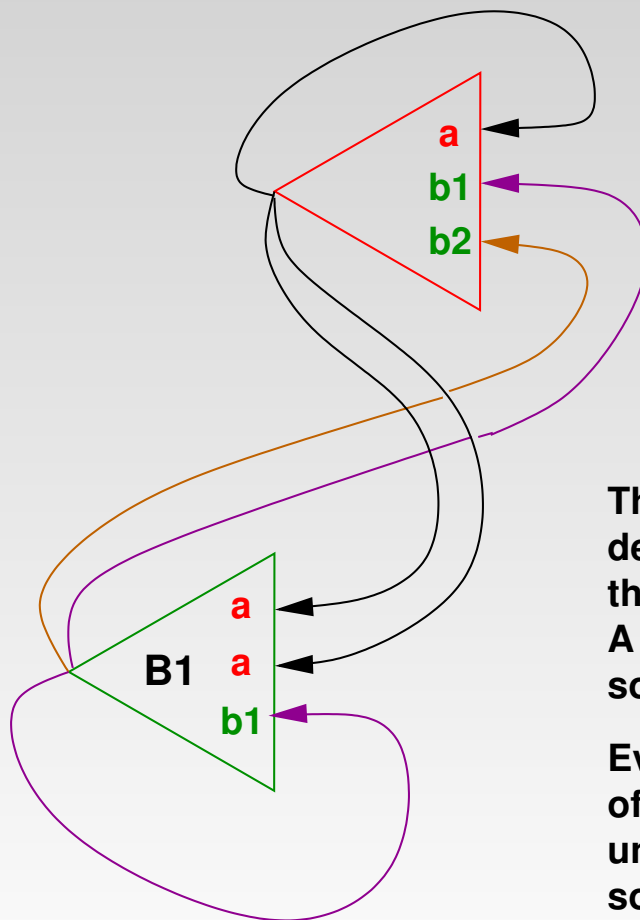
Repatching 3

Type A: red

Type B: green



Minimal network



$$A' = F(A;A,B,B)$$

$$B' = G(B;A,A,B)$$

Note: $F(x;y,u,v) \neq F(x;y,v,u)$

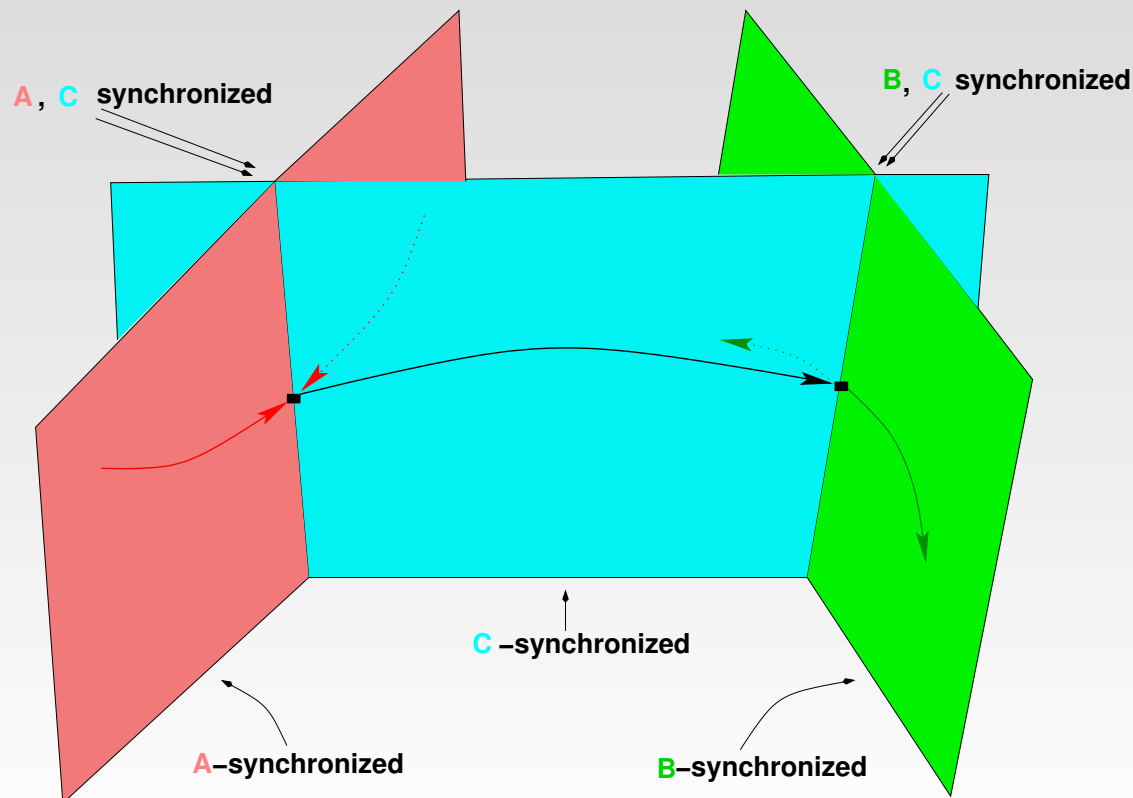
but $G(u;y,z,u) = G(u;z,y,u)$.

This coupled cell system determines dynamics on the invariant subspace of A and B synchronized solutions.

Every solution $(A(t),B(t))$ of this system extends uniquely to a synchronized solution of the original system and conversely.

Heteroclinic cycles

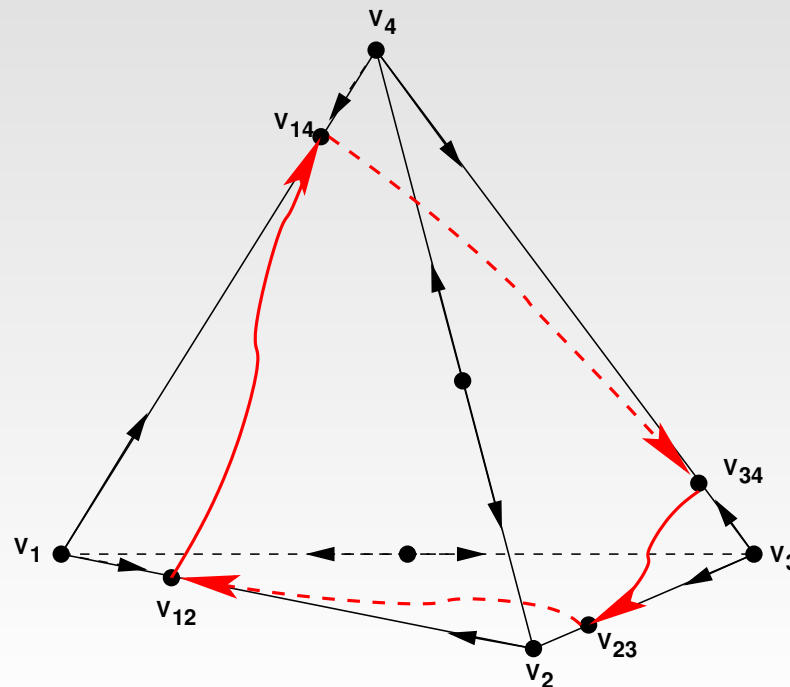
We describe results on the existence of heteroclinic cycles in (asymmetric) coupled cell systems. First, observe that we can get stable connections between **A, B** synchronised states and **B, C** synchronised states.



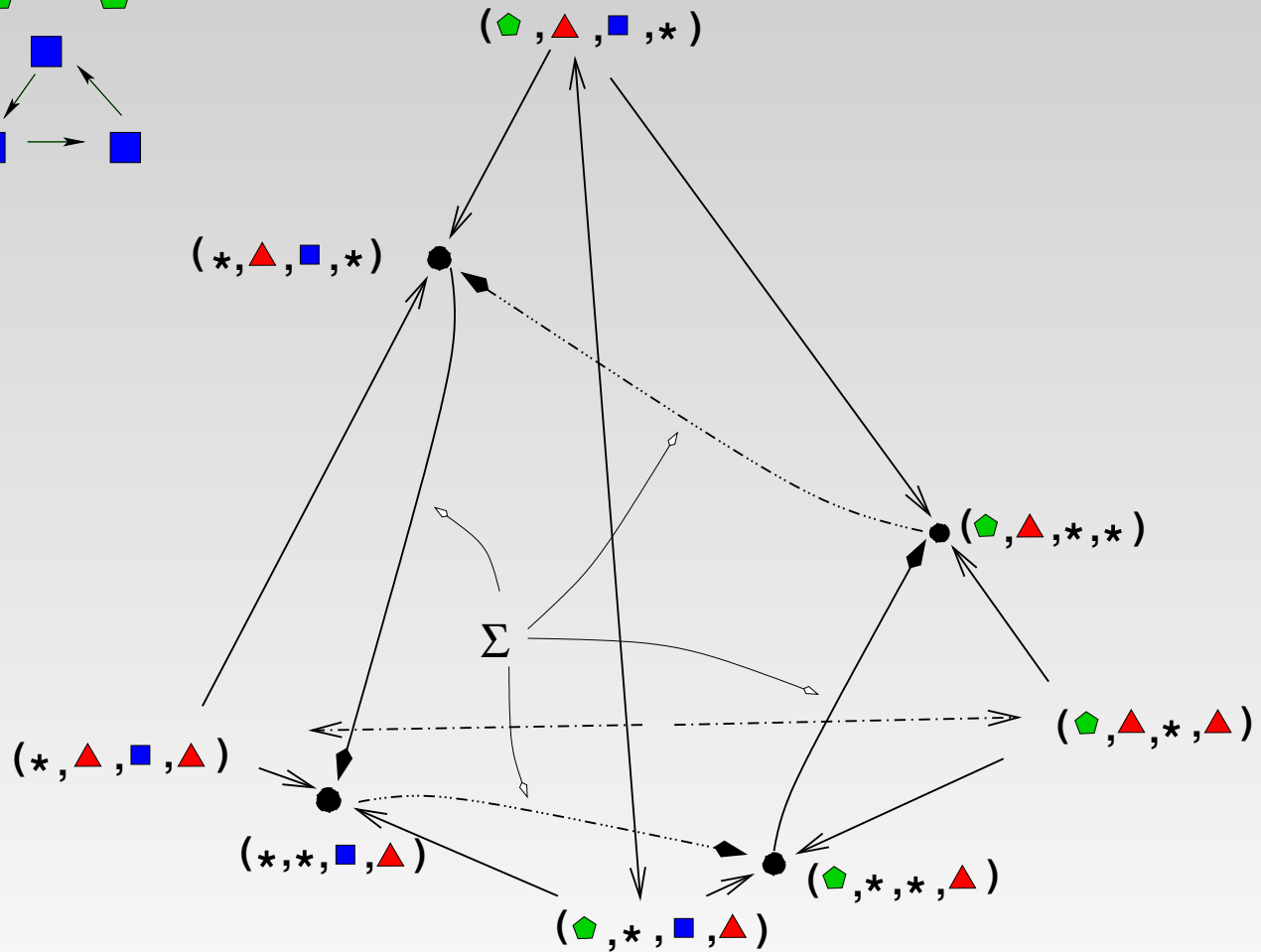
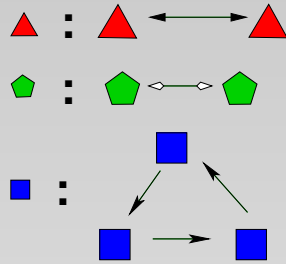
Heteroclinic cycles ctd

This gives us a way of constructing robust heteroclinic cycles between groups of synchronised cells.

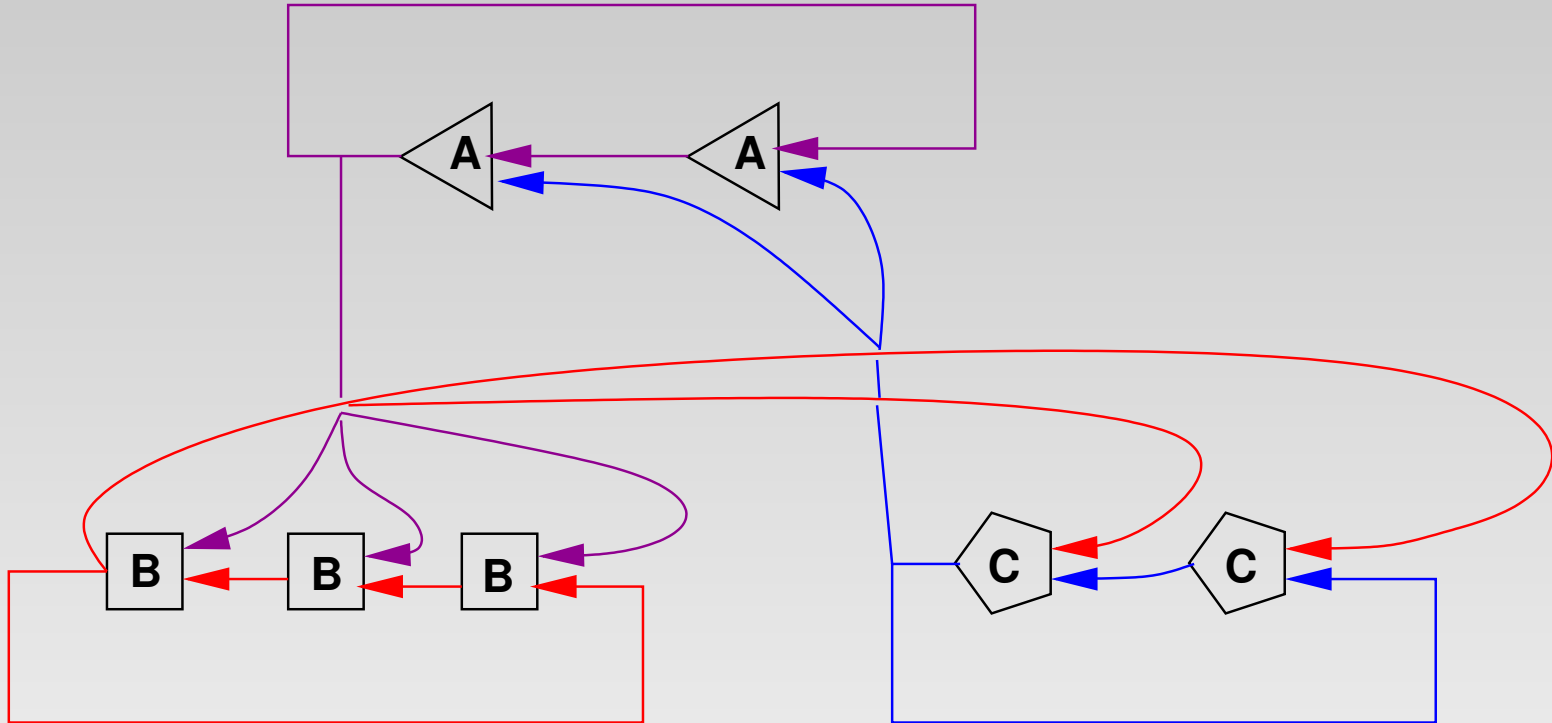
Based on our earlier models we can make a transition between edge, face, ... heteroclinic cycles and corresponding heteroclinic cycle between synchronised states.



Heteroclinic cycles ctd



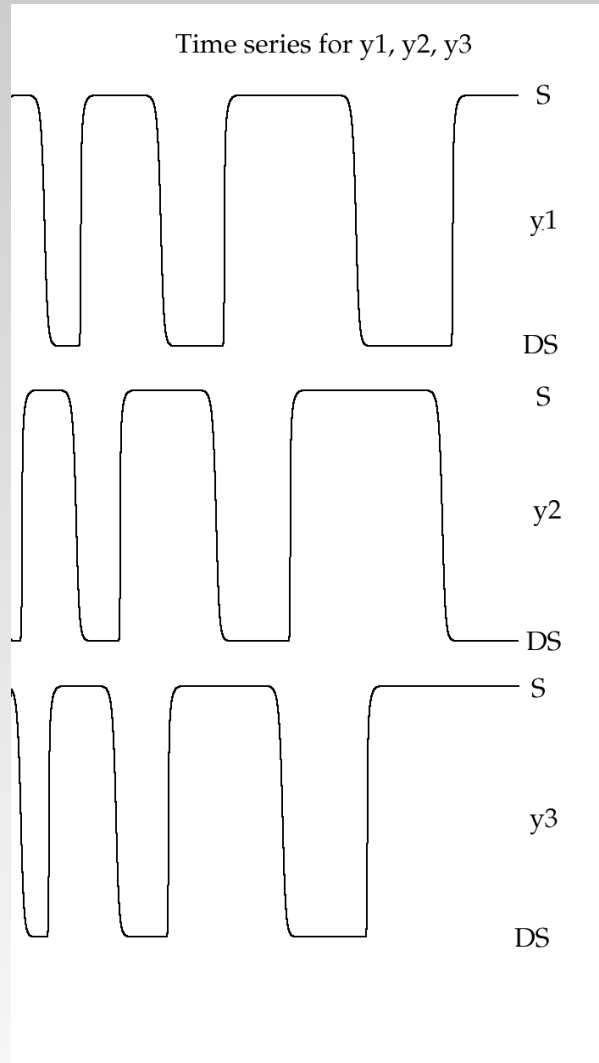
Cycles between synchronous clusters



Here we might expect to see a cycle of the form

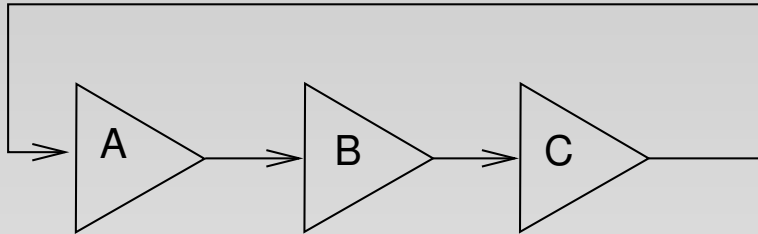
$$\dots \rightarrow AB \rightarrow BC \rightarrow CA \rightarrow AB \rightarrow \dots$$

Example

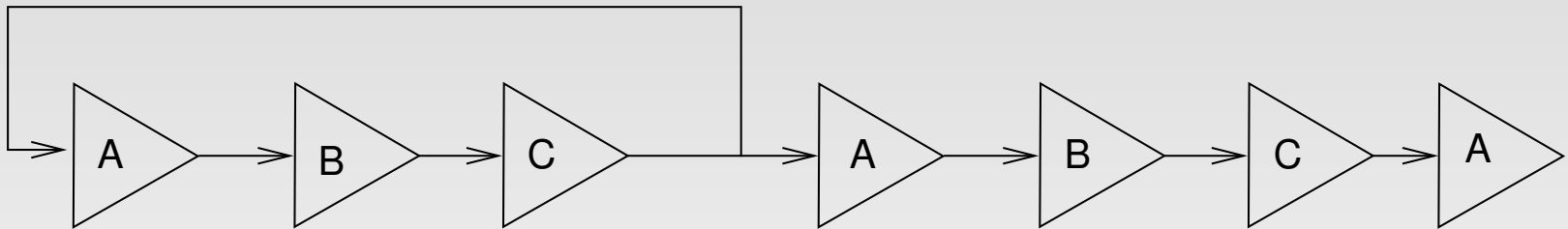


A cycle between 3 pairs of synchronous states

Embedded cycles

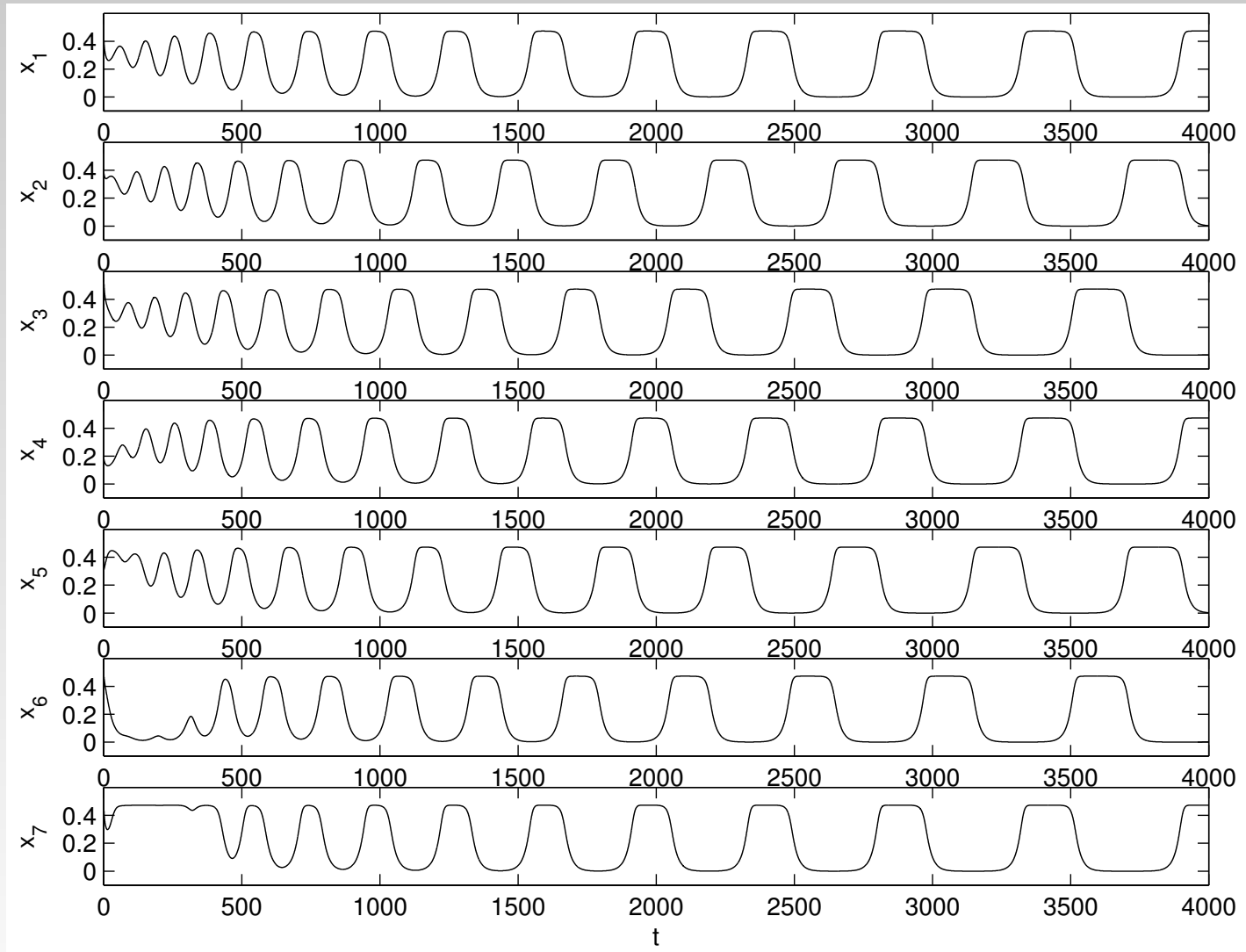


SLF system with a heteroclinic cycle

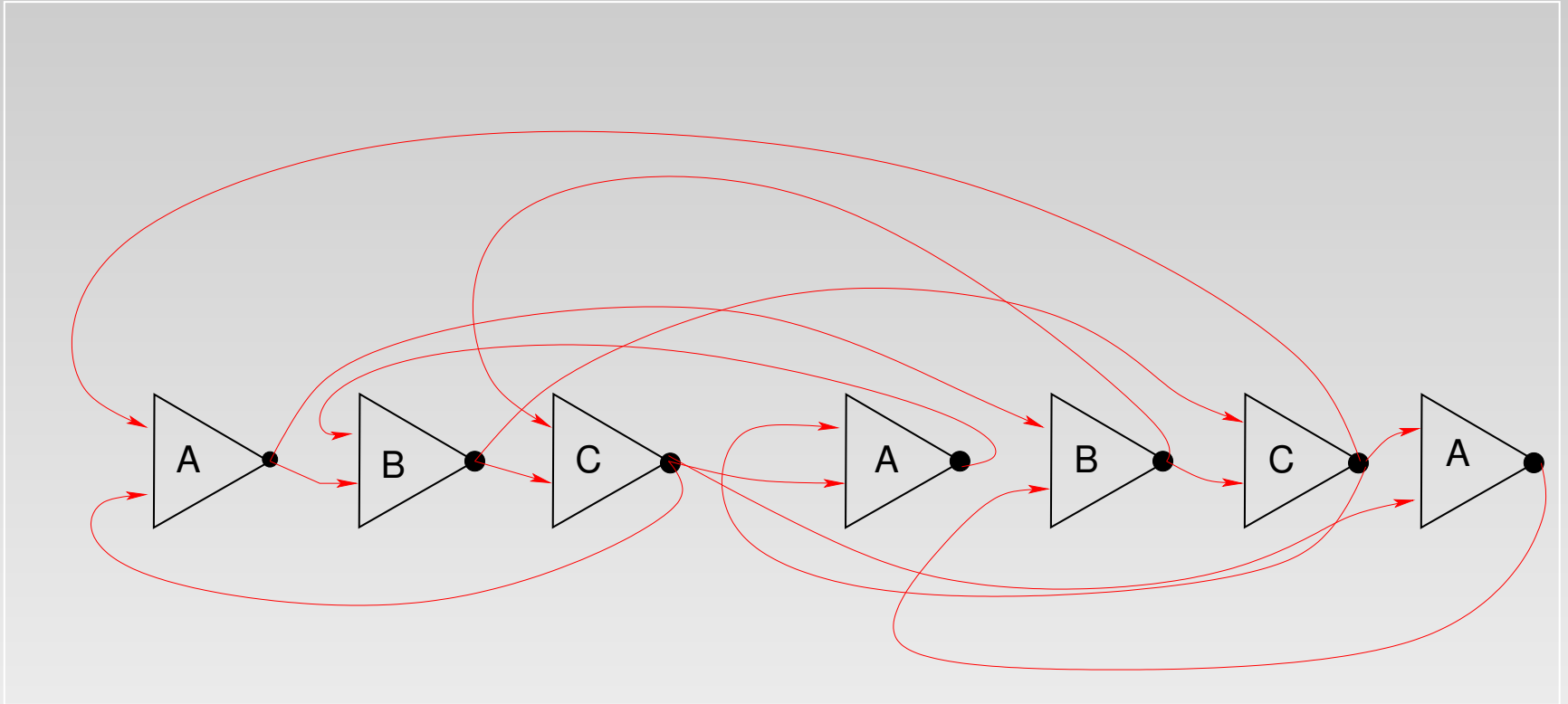


Asymmetric coupled cell system which admits a heteroclinic cycle.

Dynamics

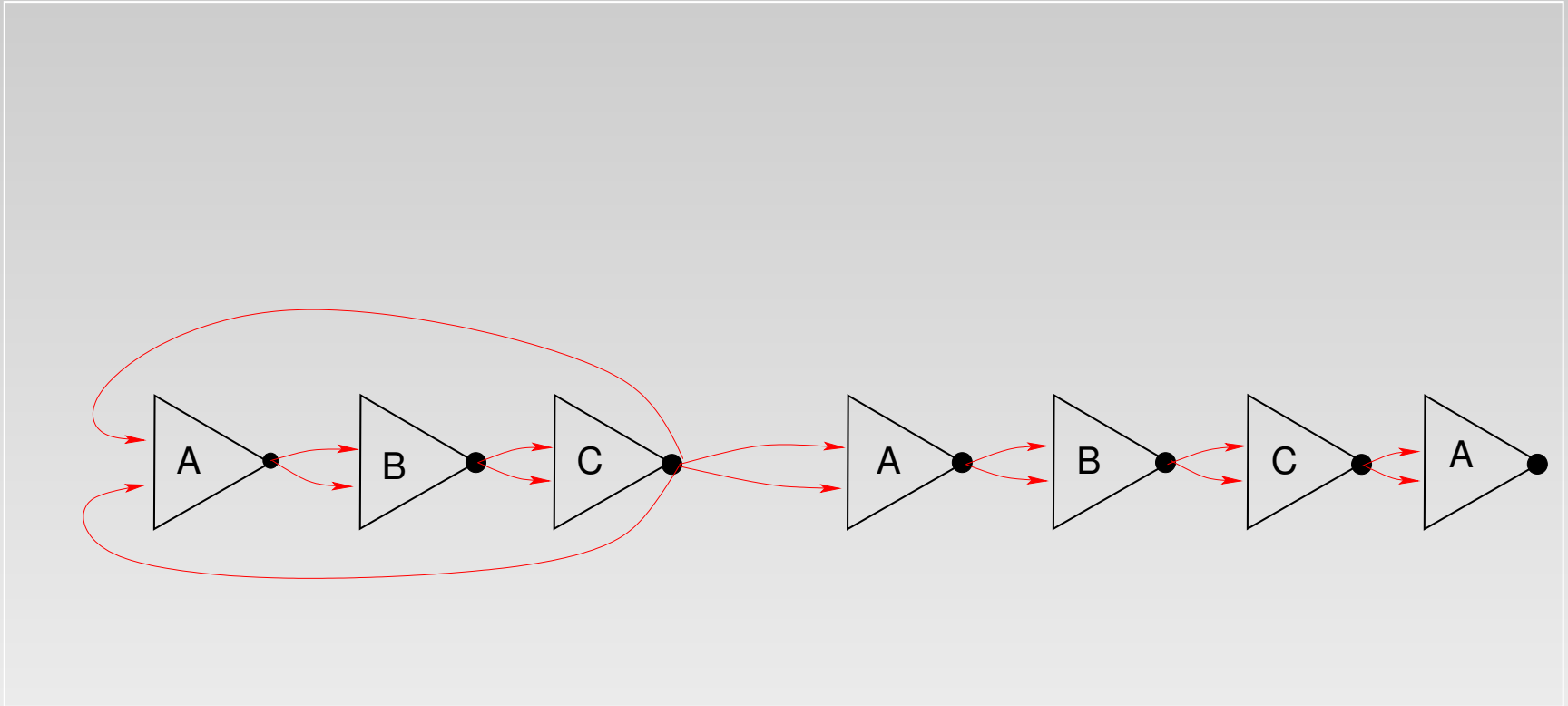


Candidate cycle



This system repatches – preserving **A, B, C** synchronous subspace – to

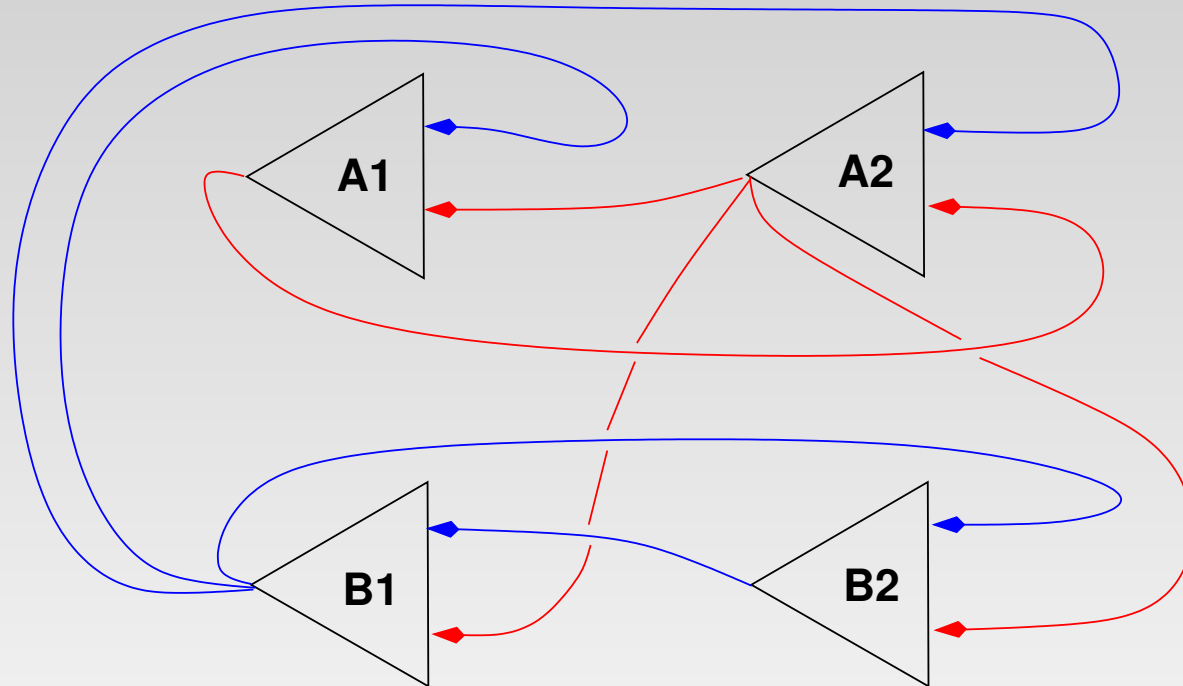
Repatched system



This system admits a heteroclinic cycle. Does the original system? Yes. Stability properties? What other combinatorial ways are there of generating/analysing heteroclinic cycles in coupled cell systems?

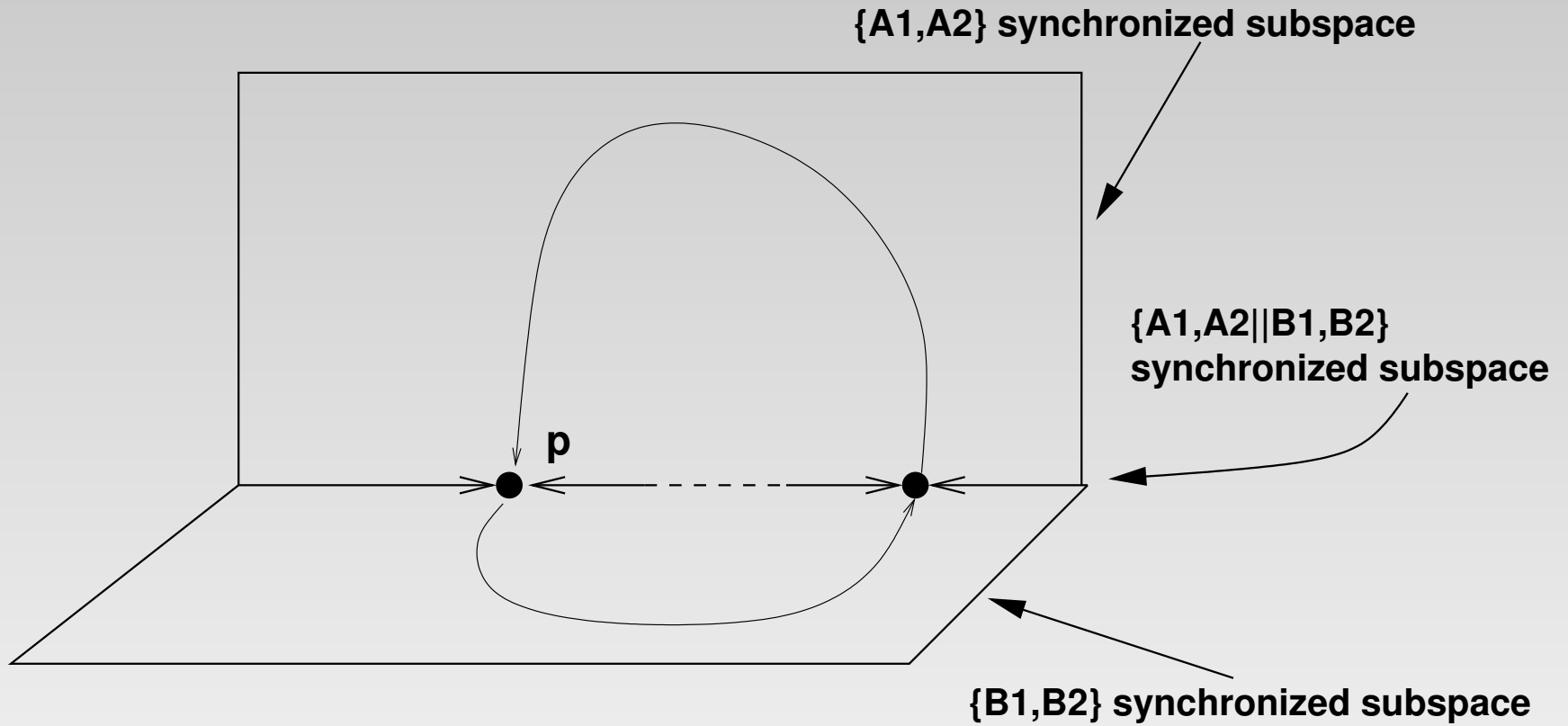
Simplest example?

$$\begin{aligned}A_1' &= F(A_1; A_2, B_1), \\A_2' &= F(A_2; A_1, B_1), \\B_1' &= F(B_1; A_2, B_2), \\B_2' &= F(B_2; A_2, B_1).\end{aligned}$$



Synchronized states: $\{A_1, A_2\}$, $\{B_1, B_2\}$, $\{A_1, A_2 || B_1, B_2\}$, $\{A_1, A_2, B_1, B_2\}$

Possible heteroclinic cycle



Some analysis

Suppose we have one-dimensional cell dynamics. The vector field F governing the evolution of the cells is

$$x'_1 = f(x_1; x_2, y_1),$$

$$x'_2 = f(x_2; x_1, y_1),$$

$$y'_1 = f(y_1; x_2, y_2),$$

$$y'_2 = f(y_2; x_2, y_1).$$

Here x variables correspond to **A** cells, y variables to **B** cells, and $f : \mathbb{R}^3 \rightarrow \mathbb{R}$ is a smooth function (no constraints).

If the system has an equilibrium at $\mathbf{p} = (0, 0, 1, 1)$, then $f(0; 0, 1) = f(1; 0, 1) = 0$.

Local representation of F

Near \mathbf{p} we consider f of the form

$$f(u; v, w) = (u - 1)^2 g_1(u; v, w) + u^2 g_2(u; v, w),$$

where g_1, g_2 are smooth.

Suppose that

$$\begin{aligned} g_1(u, v, w) &= \alpha u + \beta v + \gamma(w - 1), \\ g_2(u, v, w) &= a(u - 1) + bv + c(w - 1), \end{aligned}$$

where α, β, γ and a, b, c are constant near $(0, 0, 1)$ and $(1, 0, 1)$ respectively. We compute the linearisation $DF(\mathbf{p})$ at \mathbf{p} .

Jacobian Matrix

We find that

$$DF(\mathbf{p}) = \begin{pmatrix} \alpha & \beta & \gamma & 0 \\ \beta & \alpha & \gamma & 0 \\ 0 & b & a & c \\ 0 & b & c & a \end{pmatrix}.$$

This matrix has eigenvalues given by

Eigenvalues of Jacobian

1.

$$\frac{\alpha + \beta + a + c \pm \sqrt{(a + c - \alpha - \beta)^2 + 4\gamma b}}{2}$$

Eigenspace: $\{A1, A2 \parallel B1, B2\}$.

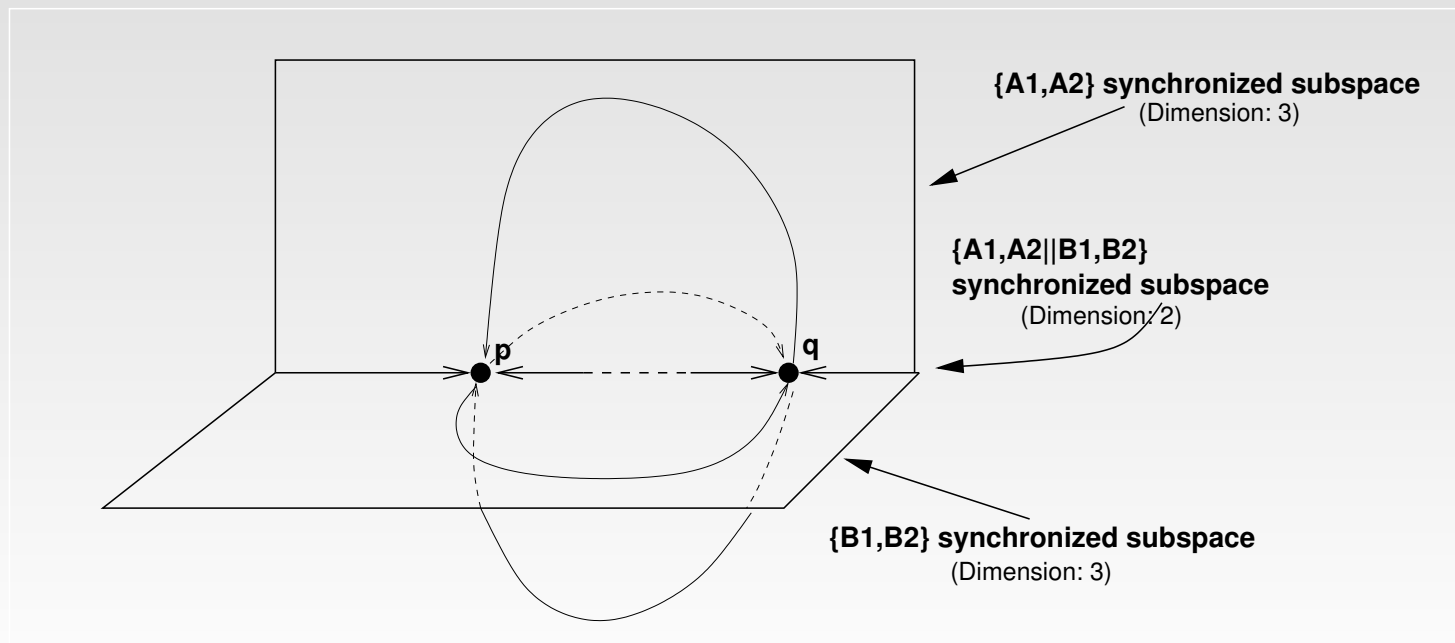
2. $\alpha - \beta$. (Eigenspace lies in $\{B1, B2\}$.)

3. $a - c$. (Eigenspace lies in $\{A1, A2\}$.)

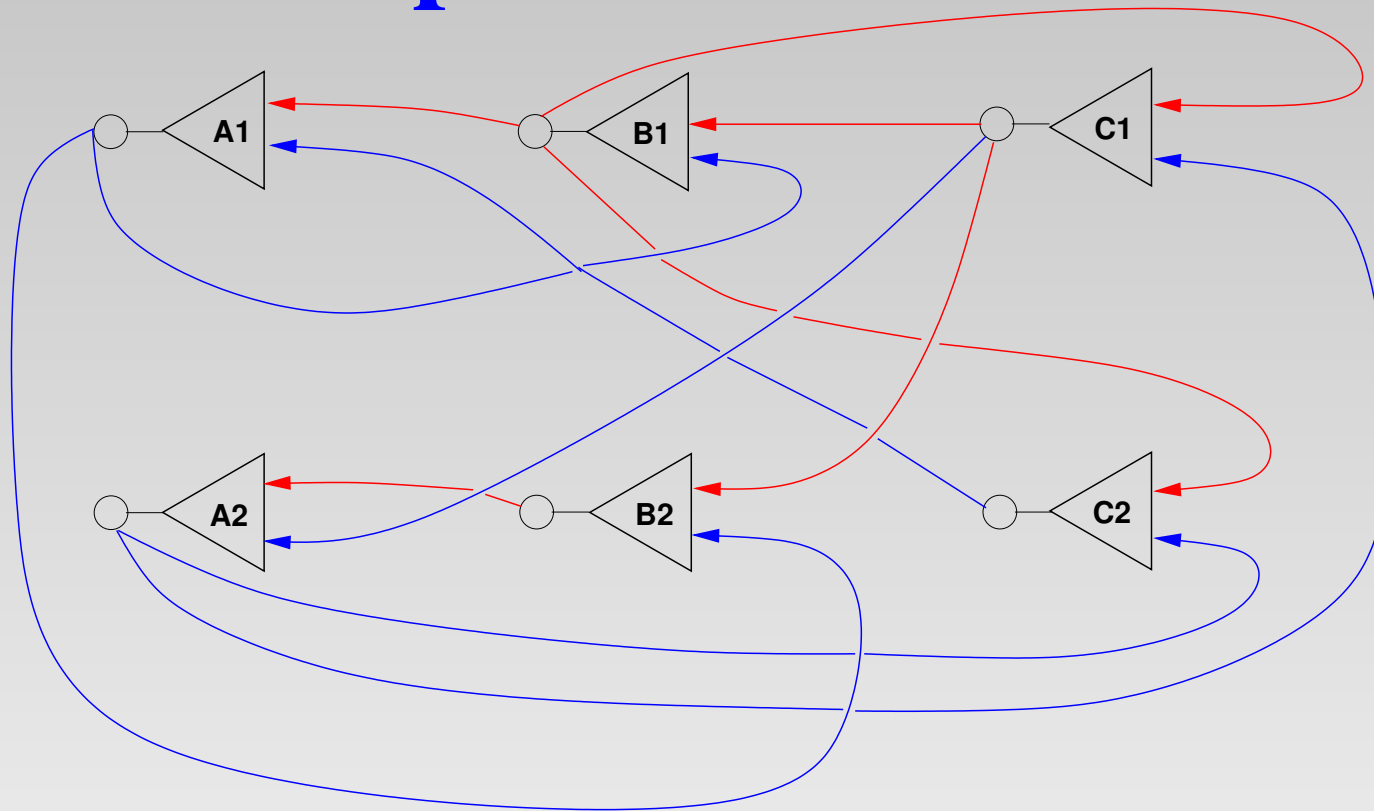
If $\alpha, \beta, a, c < 0$, $\alpha > \beta$, $a < c$, and $\gamma b < 0$, then \mathbf{p} is of index three and $W^u(\mathbf{p}) \subset \{B1, B2\}$.

Note that for sufficiently negative γb we may require the pair of eigenvalues for $\{A1, A2 \parallel B1, B2\}$ to be complex.

We may similarly choose f near $\mathbf{q} = (1, 1, 2, 2)$ so that \mathbf{p} is of index three with $W^u(\mathbf{q}) \subset \{A1, A2\}$. Finally, we can choose f on the complement of neighbourhoods of \mathbf{p}, \mathbf{q} so that there are a pair connections from \mathbf{p} to \mathbf{q} in $\{B1, B2\}$ and a pair of connections from \mathbf{q} to \mathbf{p} in $\{A1, A2\}$.

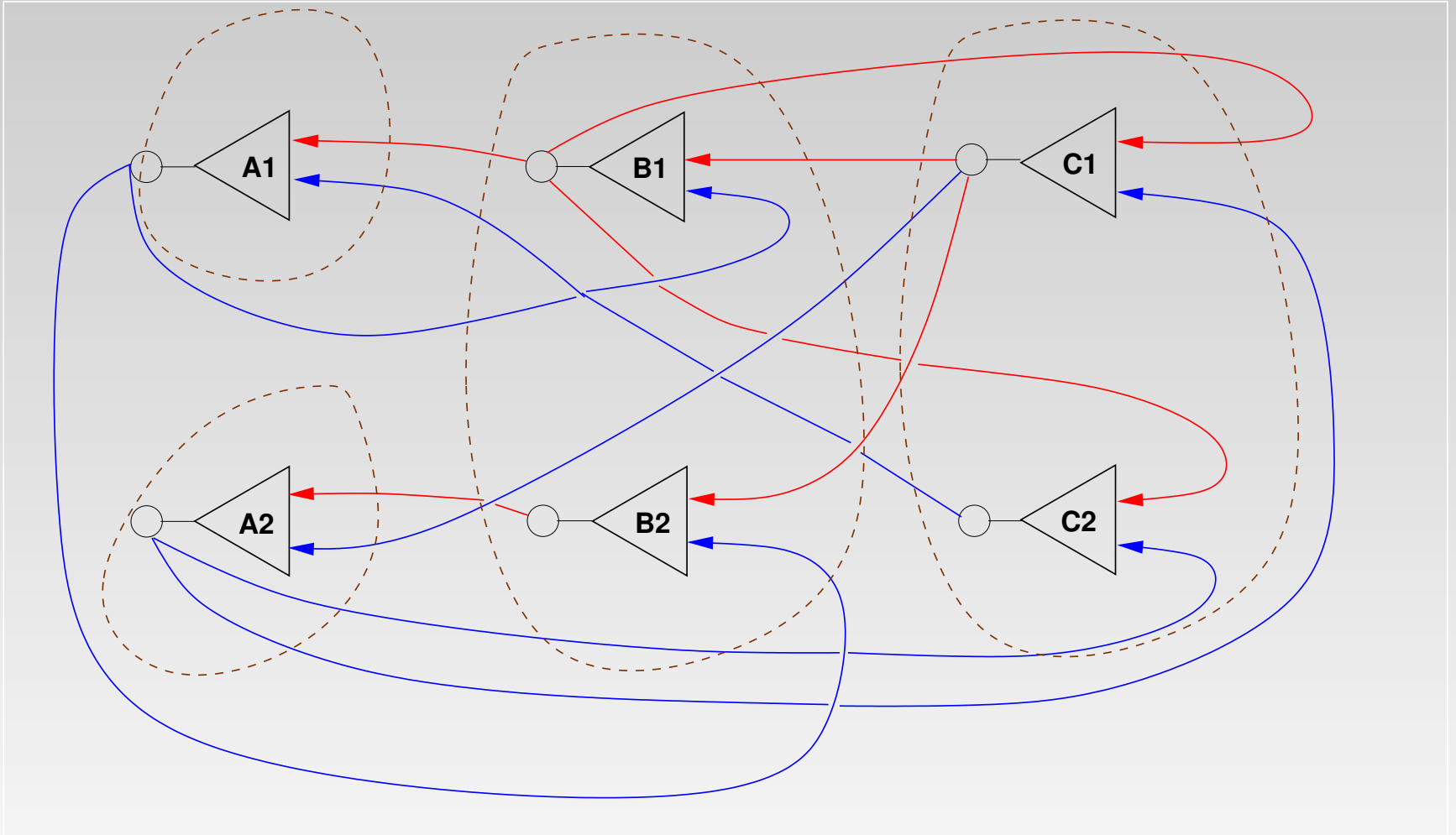


A final example

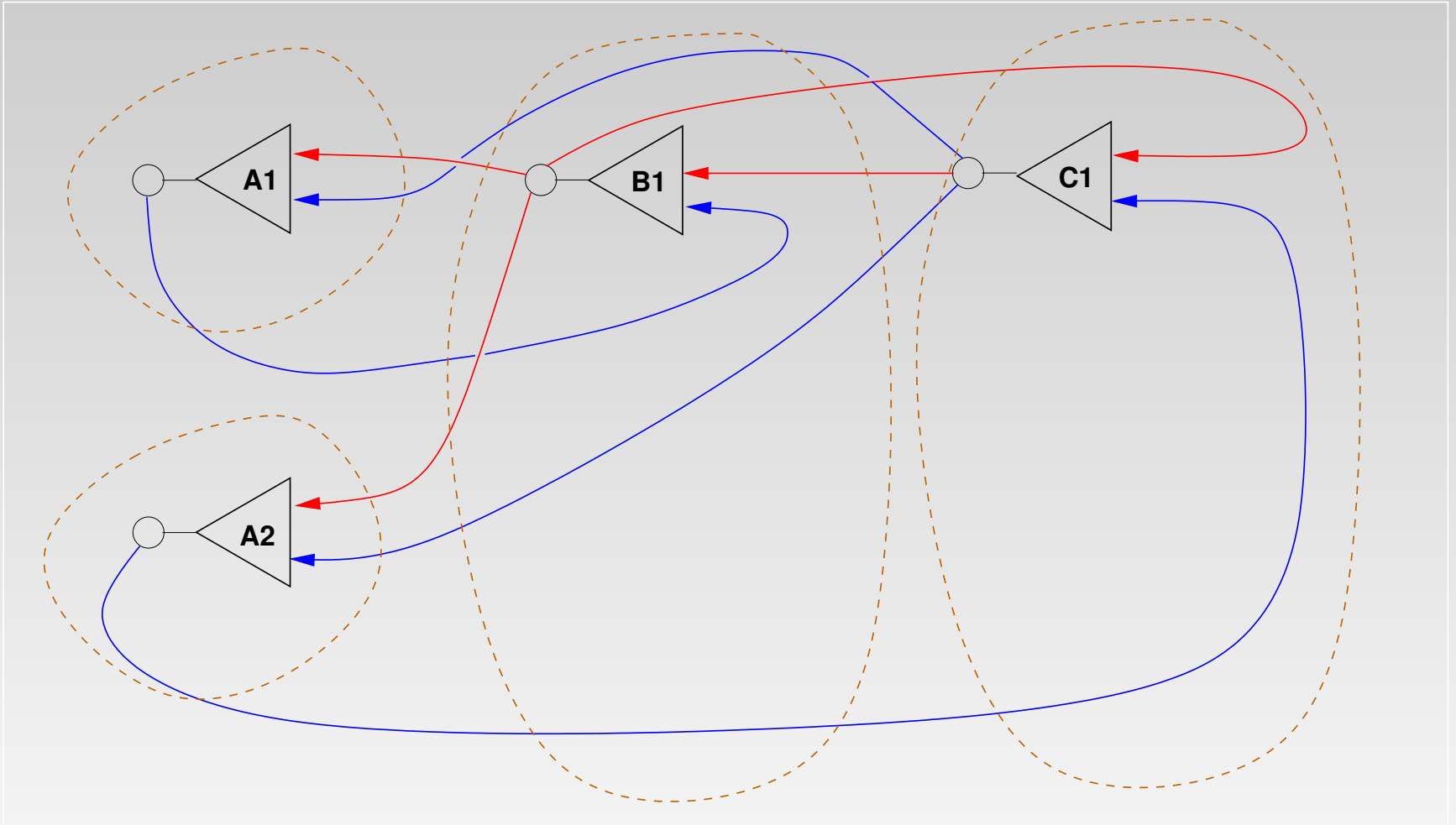


Asymmetric system of identical cells. Synchronous states $\{B1, B2\}$, $\{C1, C2\}$, $\{B1, B2 \parallel C1, C2\}$, $\{A1, A2 \parallel B1, B2 \parallel C1, C2\}$ and trivial synchronised state. In this case one can show there is a 1-dimensional heteroclinic cycle $B \rightarrow BC \rightarrow C \rightarrow BC$.

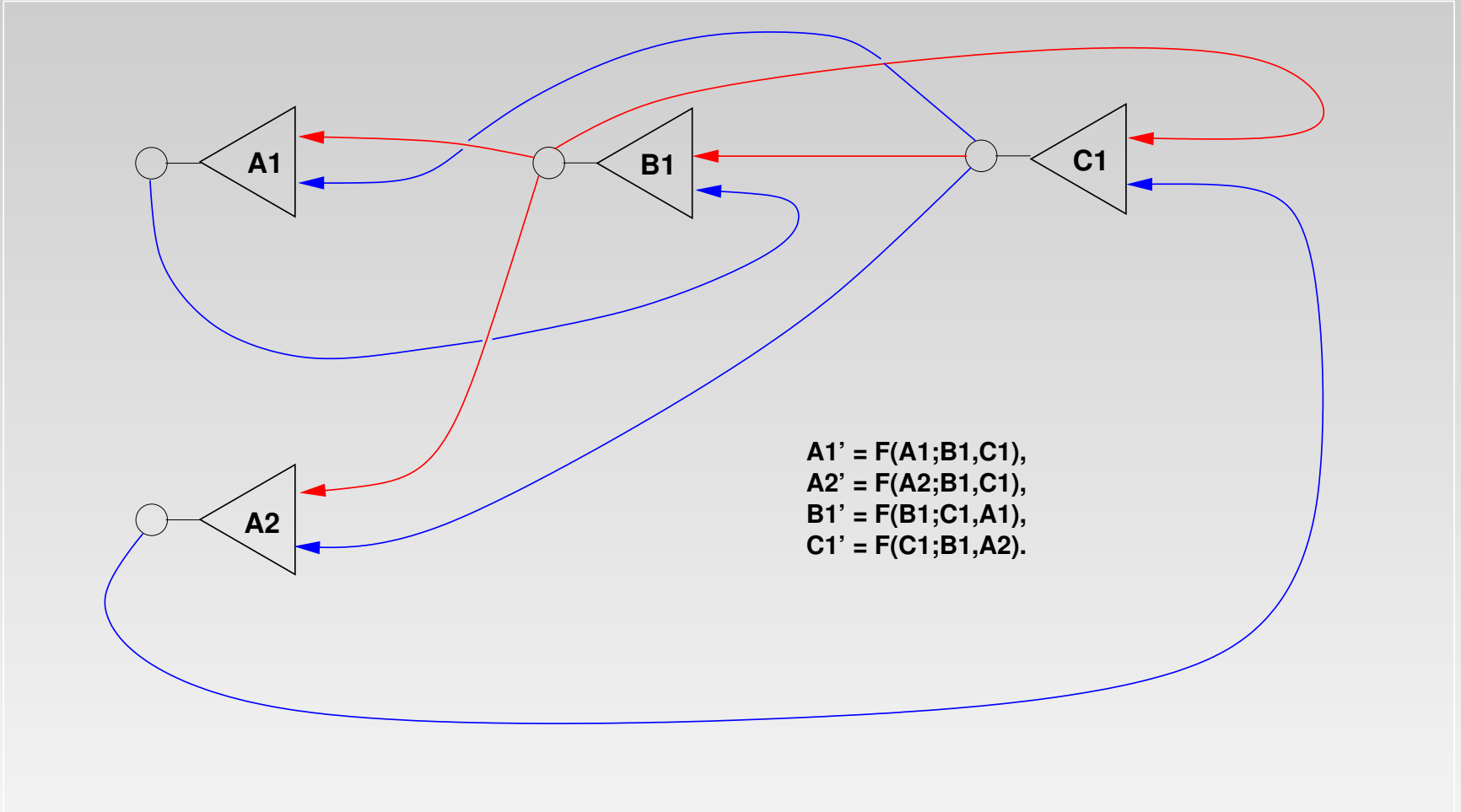
Dynamics on $\{B1, B2 \parallel C1, C2\}$ subspace



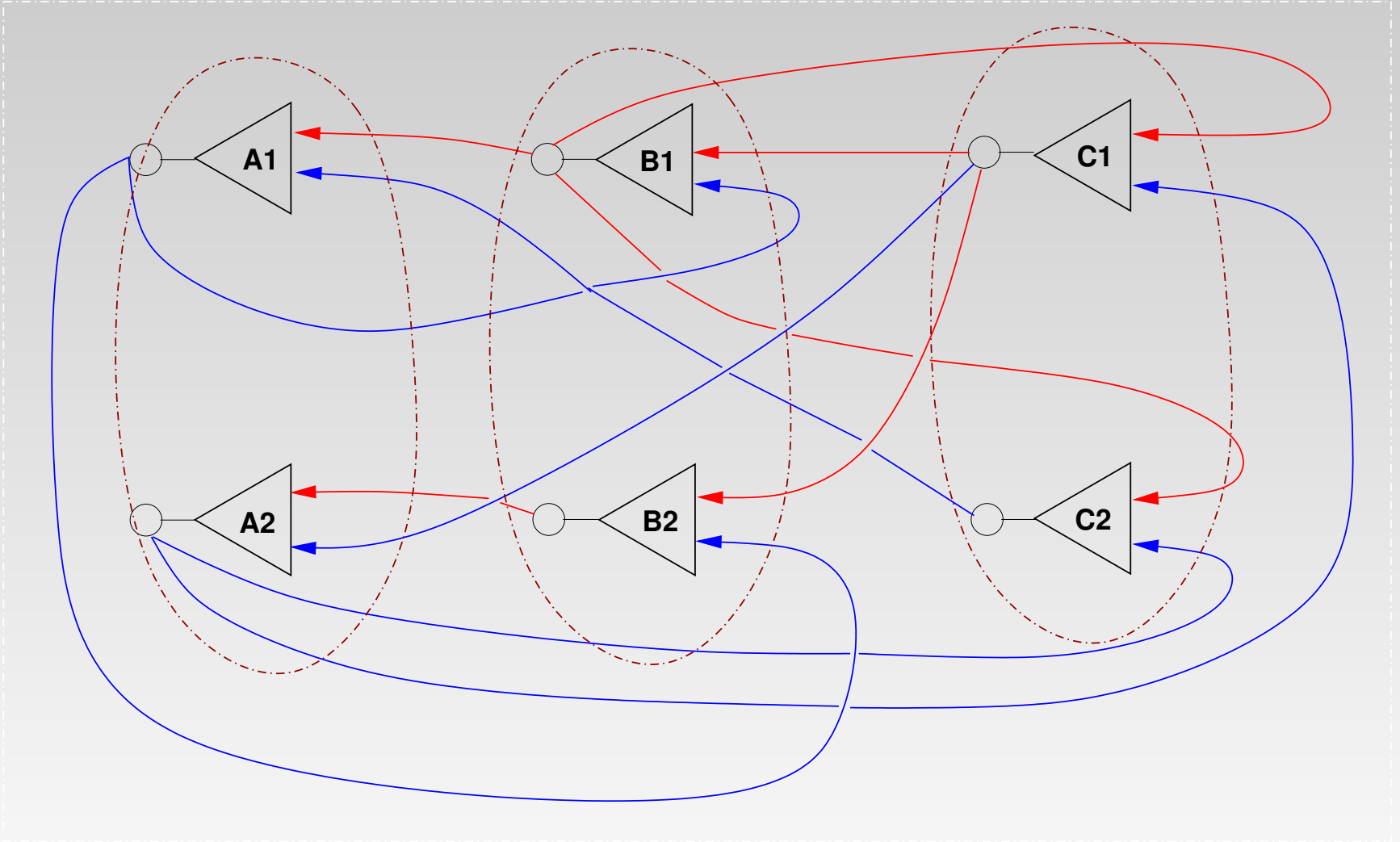
Dynamics on $\{B1, B2 \parallel C1, C2\}$ subspace



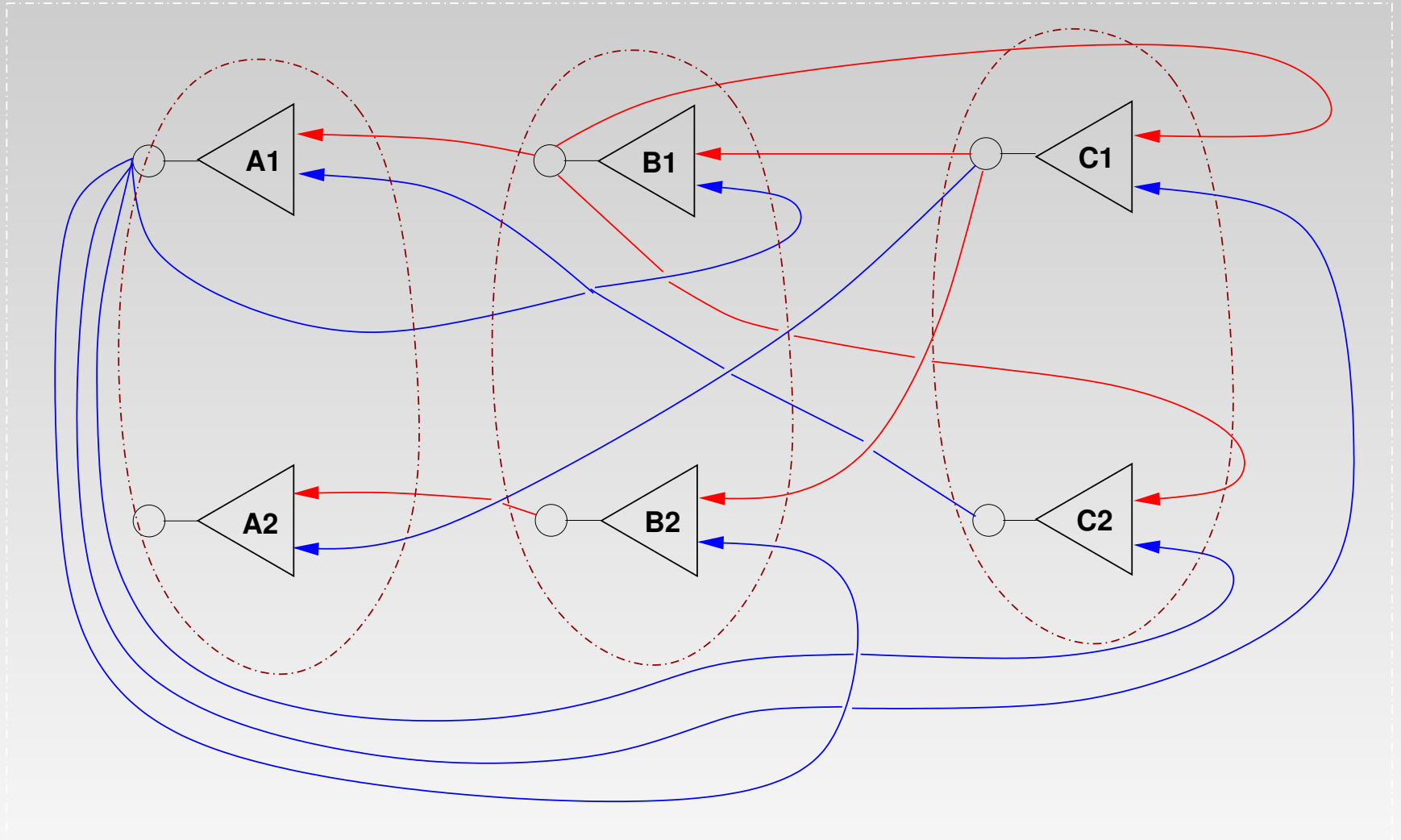
Dynamics on $\{B1, B2 || C1, C2\}$ subspace



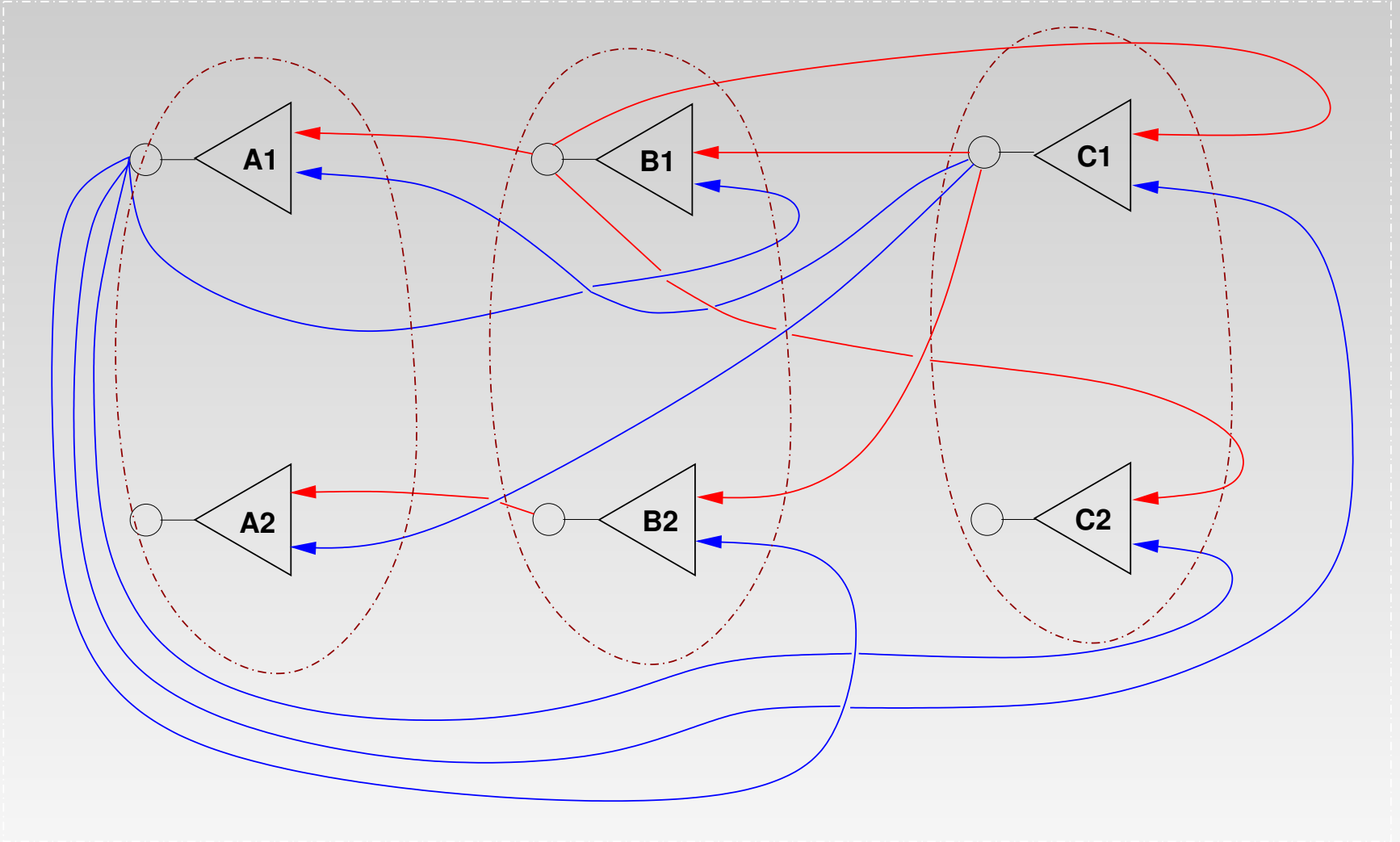
Dynamics on $\{A1, A2 \parallel B1, B2 \parallel C1, C2\}$ subspace



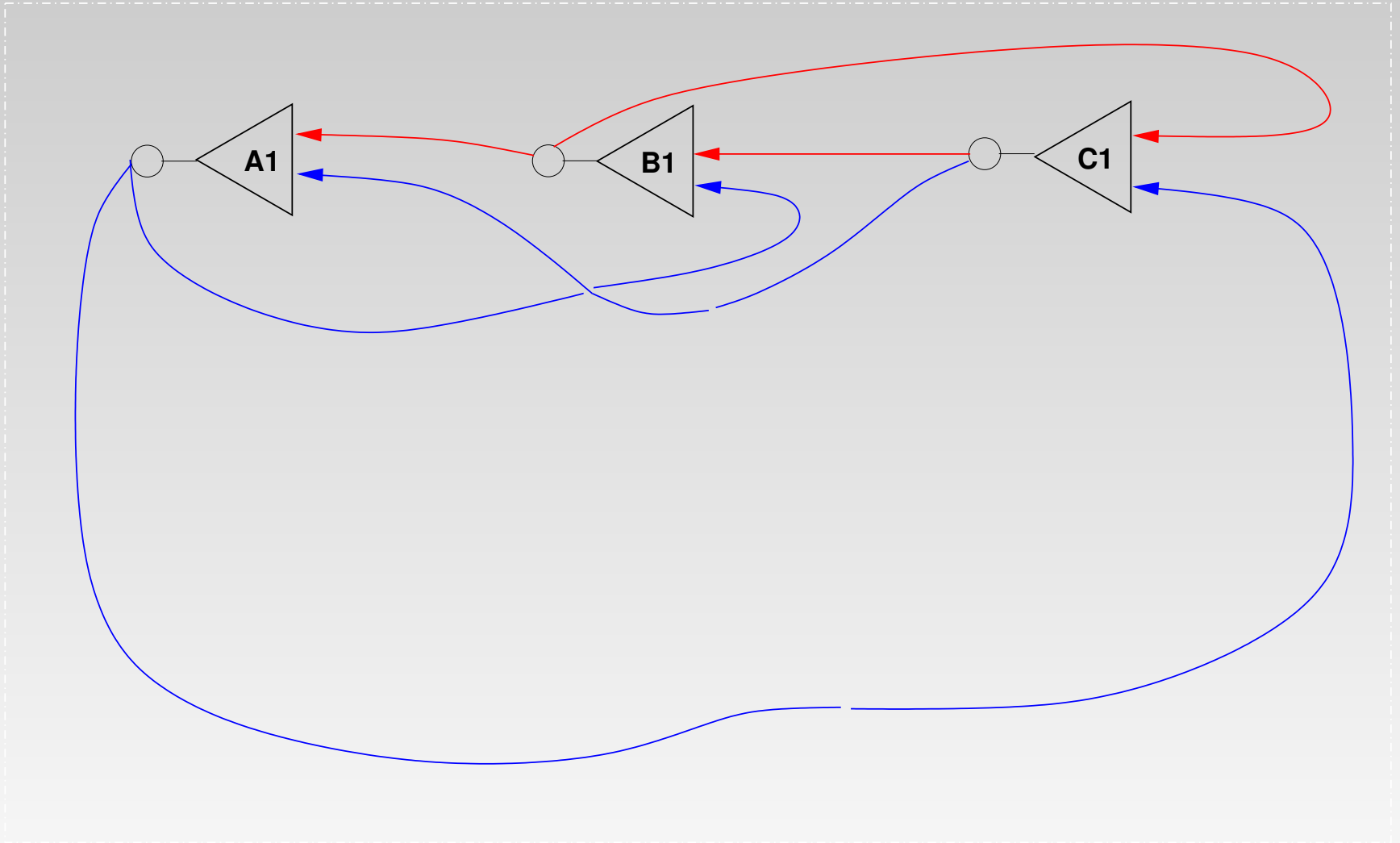
Dynamics on $\{A1, A2 \parallel B1, B2 \parallel C1, C2\}$ subspace



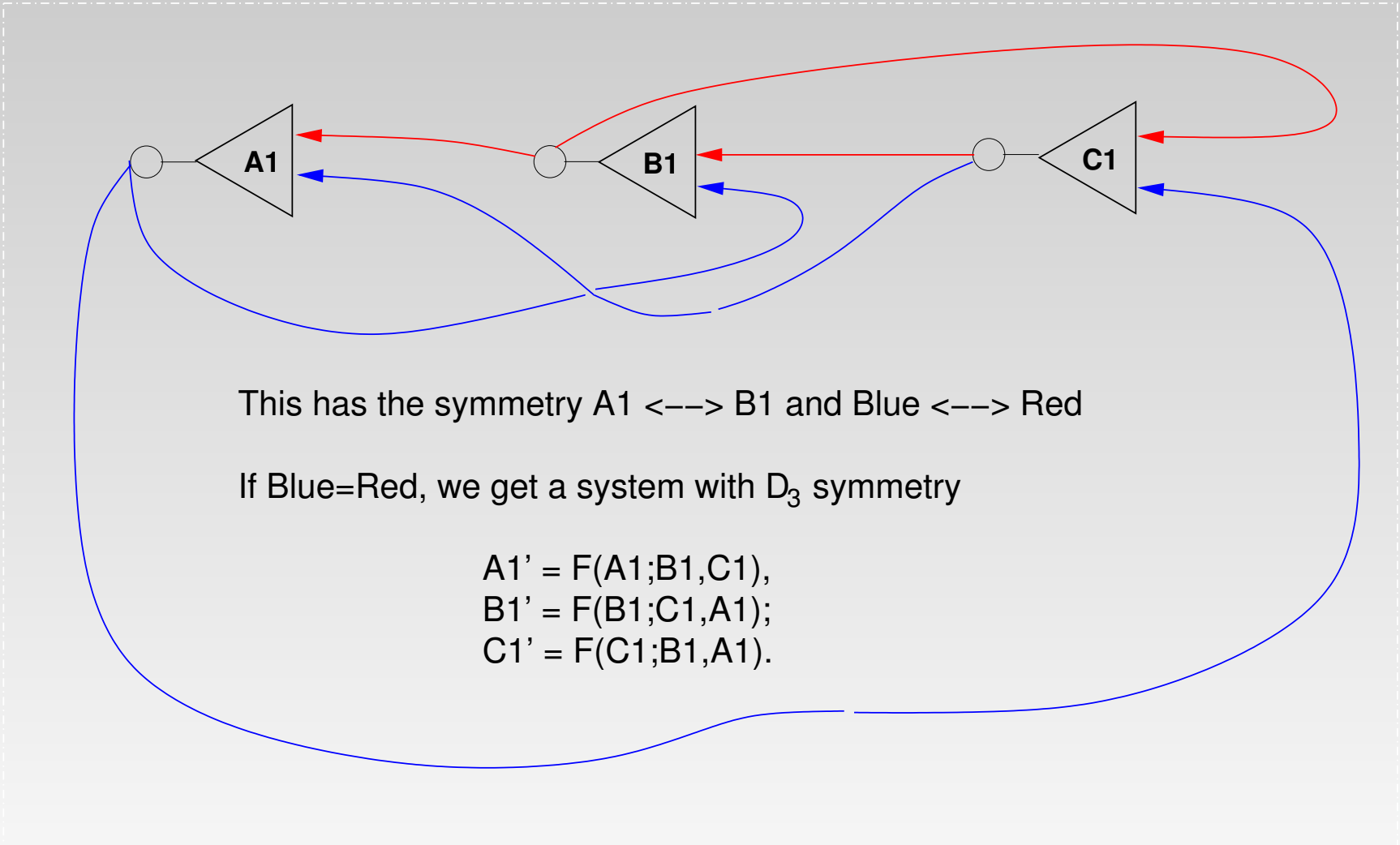
Dynamics on $\{A1, A2 \parallel B1, B2 \parallel C1, C2\}$ subspace



Dynamics on $\{A1, A2 \parallel B1, B2 \parallel C1, C2\}$ subspace



Dynamics on $\{A1, A2 \parallel B1, B2 \parallel C1, C2\}$ subspace



Invariant subspaces

Suppose that a coupled cell system $\mathcal{C} = \{A_1^i, B_1^j, \dots\}$ has an invariant subspace \mathbf{E} corresponding to a class of synchronised solutions.

We have an associated partition $\mathcal{P} = \{\mathcal{C}_k\}$ of \mathcal{C} defined by grouping synchronised cells.

Unsynchronised cells define singletons in the partition. Groups of cells of identical type but different synchronisation lie in different elements of the partition. Each element of partition will consist of cells of the same type.

Invariant subspaces

LEMMA

Let $\mathcal{C}_i, \mathcal{C}_j \in \mathcal{P}$ (they may be the same). Let α be the type of cells in \mathcal{C}_i . Let $d \in \mathcal{C}_j$. The total number of inputs into d from cells in \mathcal{C}_i depends only on i, j .

Proof. If there is variation in the number then we can choose asymmetric inputs for the cells in \mathcal{C}_j and so \mathbf{E} could not be an invariant subspace comprised of synchronous solutions...

[If not, we can find a pair of cells in \mathcal{C}_j which receive a different number of inputs from cells of type α in some \mathcal{C}_k ($k \neq i$). Choose the initial state of the cells in \mathcal{C}_k to be different from the initial states of the cells in \mathcal{C}_i .]

Repatching rules

We call a partition of \mathcal{C} satisfying the conditions of the Lemma for all i, j a *balanced* partition.

If $\mathcal{P} = \{\mathcal{C}_k\}$ is a balanced partition of \mathcal{C} , then there is an associated invariant subspace of synchronous solutions. Clusters are given explicitly by the partition.

The conditions of the lemma give repatching rules. Fix $\mathcal{C}_i, \mathcal{C}_j \in \mathcal{P}$. We can permute outputs of cells in \mathcal{C}_i that go to \mathcal{C}_j without restriction. The resulting network still has an invariant subspace with same cells synchronised as the original network. This gives rise to a unique minimal network defining the dynamics of the class of synchronised solutions. ◀