Product dynamics for homoclinic attractors

By Peter Ashwin[1] and Michael Field[2]

([1]) *Department of Mathematical Sciences, Laver Building, University of Exeter, Exeter EX4 4QE, UK.* ([2]) *Department of Mathematics, University of Houston, Houston, TX 77204-3008, USA*

Heteroclinic cycles may occur as structurally stable asymptotically stable attractors if there are invariant subspaces or symmetries of a dynamical system. Even for cycles between equilibria, it may be difficult to obtain results on the generic behaviour of trajectories converging to the cycle. For more complicated cycles between chaotic sets, the nontrivial dynamics of the 'nodes' can interact with that of the 'connections'. This paper focuses on some of the simplest problems for such dynamics where there are direct products of an attracting homoclinic cycle with various types of dynamics. Using a precise analytic description of a general planar homoclinic attractor, we are able to obtain a number of results for direct product systems.

We show that for flows that are a product of a homoclinic attractor and a periodic orbit or a mixing hyperbolic attractor, the product of the attractors is a minimal Milnor attractor for the product. On the other hand, we present evidence to show that for the product of two homoclinic attractors, typically only a small subset of the product of the attractors is an attractor for the product system.

Keywords: Milnor attractor, heteroclinic cycle, connection selection

1. Introduction

A prerequisite for understanding the dynamical behaviour of a coupled system is first to understand the dynamics of the uncoupled (that is, product) system.

As an illustration, consider two systems with attracting limit cycles L_1 and L_2 of periods P_1 and P_2 respectively. In all cases one can show that the product $L_1 \times L_2$ is a Milnor attractor for the product system. In the absence of any symmetries, the ratio P_1/P_2 of the periods is typically irrational and the product is a *minimal* Milnor attractor on which there is quasiperiodic flow. For a zero measure set of P_1/P_2, namely rational values, $L_1 \times L_2$ can be decomposed into smaller attractors consisting of periodic orbits. This observation motivates the study of quasiperiodic behaviour in more general cases where there is coupling between the systems. In particular, it suggests that finding conjugating transformations to a product system is a way to understand the persistence of quasiperiodic behaviour in coupled systems (Broer *et al.* 1990).

In this paper we examine the dynamics of products when the first factor is an attracting *homoclinic* cycle and the second factor is either an attracting limit cycle, or a hyperbolic flow or another homoclinic attractor. The questions we address are whether the typical dynamics of the product system will have an attractor that is the product of the attractors of the factors, and which attractors are minimal. We regard this work as a first step towards the analysis of more complicated systems such as skew extensions of a homoclinic attractor. However, it turns out that analysis of product systems containing a homoclinic attractor as a factor is surprisingly delicate and non-trivial. For this reason, we restrict attention in this work to establishing basic results for product systems and do not consider skew products or more general coupled systems with a homoclinic 'factor'. Note that Stone and Holmes (1989) have results on chaotic or random forcing of homoclinic attractors, but their work does not enable computation of Milnor attractors for direct product systems.

We remark that although homoclinic and heteroclinic attractors only appear at codimension one or higher for unconstrained dynamical systems, it has been known for some time that there can be robust heteroclinic attractors in systems with symmetries. This situation can occur if the connections that make up the attractor are stable within invariant subspaces. We refer to Krupa (1997) and Ashwin & Field (1999) for some studies of this effect and further references.

The paper is organised as follows. In §2 we discuss some of the definitions we use; in particular we define Milnor attractors, minimal Milnor attractors, likely limit sets and heteroclinic attractors. We also prove the following basic result which is used in the later sections.

Theorem 1.1. *Suppose that $\Phi_t = (\phi_t, \psi_t)$ is a product of C^1 flows on $X \times Y$ where $X \subset \mathbf{R}^n$ and $Y \subset \mathbf{R}^m$ are compact regions. Suppose that Λ is the likely limit set of $X \times Y$; then Λ is invariant under the $\mathbf{R}^2$-action defined by (ϕ_t, ψ_s), $(t, s) \in \mathbf{R}^2$.*

In §3 we introduce a general model for a simple homoclinic attractor $\Sigma \subset \mathbf{R}^2$. The attractor Σ will consist of a hyperbolic saddle together with one saddle connection. Dynamics will be defined on a one-sided neighbourhood H of Σ in $\mathbf{R}^2$. All points in $H \setminus \Sigma$ will have ω-limit set equal to Σ. For this model, we obtain exact expressions for the well-known algebraic slowing down of attraction to the cycle. This allows a precise and simple investigation of several product systems with homoclinic attractors. In §4 we use this model homoclinic attractor in the proof of the following result.

Theorem 1.2. *Suppose that $\psi_t(x) = x + \varpi t$ is a periodic flow on $S^1 = \mathbf{R}/\mathbf{Z}$ with $\varpi \neq 0$. Let ϕ_t be a semiflow on H with homoclinic attractor Σ as in §3. Then $\Sigma \times S^1$ is a minimal Milnor attractor for the product system $\Phi_t : H \times S^1 \to H \times S^1$, $\Phi_t(u, x) = (\phi_t(u), \psi_t(x))$.*

In §5 we consider the product of a homoclinic cycle with a hyperbolic basic set. The following is typical of our results.

Theorem 1.3. *Suppose that ψ_t is a flow on a Riemannian manifold M and that X is a hyperbolic attractor for ψ_t. Let ϕ_t be a semiflow on H with homoclinic attractor Σ as in §3. Then $\Sigma \times X$ is a Milnor attractor for the product system $\Phi_t : H \times M \to H \times M$, $\Phi_t(u, x) = (\phi_t(u), \psi_t(x))$. If $\psi_t | X$ is topologically mixing, then $\Sigma \times X$ is a minimal Milnor attractor for the product system.*

In §6, we consider the product of two homoclinic attractors. One result identifies the possibilities for attractors for the product.

Theorem 1.4. *Consider a product of two systems with homoclinic attractors $\Sigma_1 \subset H_1$ and $\Sigma_2 \subset H_2$ such that Σ_i is the only attractor within H_i. Let*

$$\Sigma = (\{q_1\} \times \Sigma_2) \bigcup (\Sigma_1 \times \{q_2\}),$$

where q_i is the equilibrium point on Σ_i, $i = 1, 2$. Then either Σ or $\Sigma_1 \times \Sigma_2$ is a maximal Milnor attractor for the product.

The two dimensional invariant set $\Sigma_1 \times \Sigma_2$ is a heteroclinic network for the product dynamics. It consists of the union of all the connections joining the equilibrium (q_1, q_2) to (q_1, q_2). In the remainder of §6, we investigate the product of two particularly simple model homoclinic attractors. For this system, we are able to show that Σ is a minimal attractor for the product system. In particular, $\Sigma_1 \times \Sigma_2$ is not a Milnor attractor; rather the attractor is a sub-network containing only connections where one of the factor systems remains at an equilibrium. This result proceeds by an examination of accumulation points of the sequence $\epsilon_1 \lambda_1^n - \epsilon_2 \lambda_2^m$ where the $\lambda_i > 1$ give the rates of slowing down for Σ_i and the ϵ_i depend on initial conditions.

This result strongly suggests that the product of attracting heteroclinic cycles is typically not a Milnor attractor. In §7 we discuss the problem of 'selection of connections' for heteroclinic cycles with multi-dimensional connections, as well as some problems in extending our results to a more general setting. Specifically, we discuss some obstacles to removing assumptions about the form of the cycle or the assumption of a product structure.

2. Attractors for product systems

Consider a dynamical system defined by the continuous flow Φ_t on M, where M is a compact region in $\mathbf{R}^n$. For our intended applications, it suffices to assume that M is *forward* invariant under Φ_t (so Φ_t is a semiflow on M) and that Φ_t is the restriction of a flow defined on $\mathbf{R}^n$ (or an invariant open neighbourhood of M in $\mathbf{R}^n$).

We denote Lebesgue measure on M by ℓ. Given $x \in M$, let

$$\omega(x) = \cap_{T>0} \overline{\{\Phi_t(x) \mid t \geq T\}}, \quad \alpha(x) = \cap_{T<0} \overline{\{\Phi_t(x) \mid t \leq T\}}$$

respectively denote the ω- and α-limits set of the trajectory through x. If X is a subset of M, let $\mathcal{B}(X) = \{x \in M \mid \omega(x) \subset X\}$ denote the *basin of attraction* of X.

Definition 2.1 (Milnor 1985). A compact invariant subset $X \subset M$ is a *Milnor attractor* if

1. $\ell(\mathcal{B}(X)) > 0$.

2. For any proper compact invariant subset $Y \subset X$, $\ell(\mathcal{B}(X) \setminus \mathcal{B}(Y)) > 0$.

We say X is a *minimal attractor* if for all proper compact invariant subsets $Y \subset X$, $\ell(\mathcal{B}(Y)) = 0$.

Remarks 2.2. (1) For the systems considered in this paper, it will usually be the case that $\mathcal{B}(X)$ is a neighbourhood of X in M (but not necessarily in $\mathbf{R}^n$).
(2) Every compact invariant set X with a nonempty open basin of attraction contains a Milnor attractor. In general, the attractor will be a proper subset of X.
(3) A Milnor attractor X is minimal if and only if there is a full measure subset $B \subset \mathcal{B}(X)$ such that $\omega(x) = X$ for all $x \in B$. (Choose an increasing sequence (Y_n) of proper compact subsets of X such that $\cup_n Y_n = X$ and let $B = \mathcal{B}(X) \setminus \cup_n \mathcal{B}(Y_n)$. Observe that $\ell(\sum_n \mathcal{B}(Y_n)) = 0$ and if $\omega(x)$ is a proper subset of X then there exists n such that $\omega(x) \subset Y_n$.)
(4) It is possible to make a trivial extension to the definitions of Milnor and minimal attractors by allowing products with measure preserving transformations. This will prove useful in §5.

If Z is an invariant measurable set with $\ell(Z) > 0$, the *likely limit set* of Z (Milnor 1985) is the smallest closed invariant set that contains all ω-limit sets except for a subset of Z with ℓ-measure zero. If X is a Milnor attractor then the likely limit set of $\mathcal{B}(X)$ exists and is equal to X.

We refer to lemma 1 of Milnor (1985) for a general proof of the existence of the likely limit set. It also follows easily from results in Milnor (1985) that the likely limit set of M for a semiflow defined on a compact region $M \subset \mathbf{R}^n$ is the *maximal* attractor; that is, it is a Milnor attractor that contains all Milnor attractors in M.

Lemma 2.3. *Let Λ denote the likely limit set of Z. Then*

1. *$x \in \Lambda$ if and only if for all $\epsilon > 0$ and for all full measure subsets H of Z there exists $a \in H$ such that $B_\epsilon(x) \cap \omega(a) \neq \emptyset$ ($B_\epsilon(x)$ denotes the ϵ-ball about x).*

2. *Λ is a minimal Milnor attractor if and only if for all $x \in \Lambda$, and all $\epsilon > 0$ and for all measurable subsets H of Z of strictly positive measure,*

$$\ell(\{a \in H \mid B_\epsilon(x) \cap \omega(a) \neq \emptyset\}) > 0.$$

Proof. The first statement follows by noting that $x \notin \Lambda$ if and only if there exist $\epsilon > 0$ and a full measure subset H of Z such that $B_\epsilon(x) \cap \omega(a) = \emptyset$ for all $a \in H$. The second statement is obviously satisfied if Λ is minimal. On the other hand, suppose that Y is a compact flow invariant subset of Λ with $\mathcal{B}(Y)$ of strictly positive measure. Now apply the criterion with $B_\epsilon(x)$ disjoint from Y and $H = \mathcal{B}(Y)$. □

Proof of Theorem 1.1. We write $\Phi_{t,s} = (\phi_s, \psi_t)$. Fix $(t, s) \in \mathbf{R}^2$ and let $(x, y) \in \Lambda$. We claim that $(x', y') = \Phi_{t,s}(x, y) \in \Lambda$. In order to show this, we use the characterization of the likely limit set given by lemma 2.3(1). Choose $\epsilon > 0$; it follows by continuity that there is a $\delta > 0$ such that $\Phi_{t,s}(B_\delta(x, y)) \subseteq B_\epsilon(x', y')$. Since $\Phi_{t,s}$ is C^1, it follows that if $H' \subseteq X \times Y$ is of full measure so is $H = \Phi_{-t,-s}H'$. Hence, by lemma 2.3, there exists $(a, b) \in H$ such that $B_\delta(x, y) \cap \omega(a, b) \neq \emptyset$. Setting $(a', b') = \Phi_{t,s}(a, b)$, it follows that $(a', b') \in H'$ and $B_\epsilon(x', y') \cap \omega(a', b') \neq \emptyset$. Hence $(x', y') \in \Lambda$. □

We say a chain-recurrent compact invariant set Σ is a *heteroclinic network* (Ashwin & Field 1999) between equilibria if it has a finite proper subset $\mathcal{E}$ consisting of equilibria such that for any $x \in \Sigma$ we have $\alpha(x), \omega(x) \subset \mathcal{E}$. We refer to the points in $\mathcal{E}$ as *nodes*, and orbits in $\Sigma \setminus \mathcal{E}$ as *connections*. If there is just one node and one connection, we call Σ a *homoclinic cycle* (or just a *cycle*). More generally, we say that a heteroclinic network Σ is a *heteroclinic cycle* if there are at least two nodes and one can order the nodes cyclically so that there exists a single connecting trajectory from one node to the next, and no other connections.

We say a heteroclinic network Σ is a *heteroclinic attractor* if Σ is a Milnor attractor. This definition includes homoclinic attractors as the special case when Σ is a homoclinic cycle. A heteroclinic attractor may be a subset of a larger heteroclinic network; namely a chain-recurrent compact invariant set Σ that is not necessarily a Milnor attractor, but which contains all unstable manifolds of equilibria within Σ. Finding a heteroclinic attractor within a heteroclinic network may be a subtle problem on account of two effects: essential asymptotic stability (see Melbourne 1991), and *connection selection* where only a small subset of the network may be seen in the attractor (see Ashwin & Chossat 1997).

3. A model homoclinic attractor in $\mathbf{R}^2$

In this section we consider a simple model for an attracting homoclinic cycle. To this end, assume that we are given a smooth flow ϕ_t on the plane with a homoclinic cycle Σ connecting the origin – see figure 1.

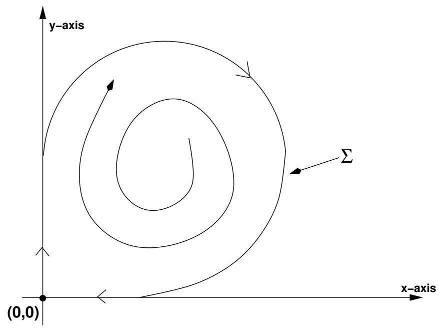

Figure 1. Vector field on the plane with homoclinic cycle

Assume that the origin is a hyperbolic saddle with associated eigenvalues $-a < 0 < b$, where

$$a > b > 0 \quad \text{and} \quad \frac{a}{b} \notin \{3/2, 2, 3, 4\}. \tag{3.1}$$

We further assume that coordinates are chosen so that the local stable manifold of the origin lies on the x-axis and the local unstable manifold of the origin lies on the y-axis. It follows from 3.1 and Samovol's version of Sternberg's linearization theorem (Samoval 1972, Belickii 1973) that (provided the vector field is at least C^7) we can make a C^3-local change of coordinates so that the vector field is linear near the origin:

$$\begin{aligned} x' &= -ax, \\ y' &= by. \end{aligned}$$

Rescaling coordinates if necessary, we suppose that these equations are valid on an open neighbourhood of the square $[0,1]^2 \subset \mathbf{R}^2$.

We start by considering the linear system. Referring to figure 2, we consider the flow of this system on a subset of the square $[0,1]^2 \subset \mathbf{R}^2$. We fix an interval $I = [0, A_0] \times \{1\}$, $A_0 < 1$, on the top side of the square. For each $(X,1) \in I$, $X \neq 0$, let $T(X)$ be the time it takes to flow from $(X,1)$ to $(1, F(X)) \in J$ – the point of

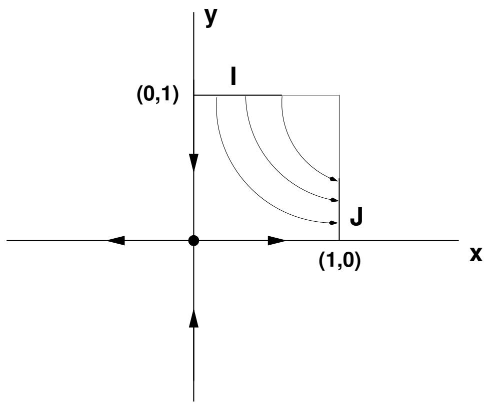

Figure 2. Linear flow near a hyperbolic fixed point at 0; I and J represent inflowing and outflowing sections.

intersection with J of the forward trajectory through $(X, 1)$. Since $a > b$, it is easy to see that $J = \{1\} \times [0, B]$, where $B < A_0$. A straightforward computation shows that

$$F(X) = X^\lambda, \;\; T(X) = \log X^{-1/b}, \;\; B = A_0^\lambda, \tag{3.2}$$

where $\lambda = a/b > 1$.

Using these expressions it is now easy to define a return map for the cycle Σ. We define α, $\beta(X)$, $\gamma(X)$ such that $(1, X)$ maps to $(X\gamma(X), 1)$ after time $\alpha + X\beta(X)$; note that β and γ are C^2 and $\gamma(0) > 0$. Combining the above gives a map $(X, 1) \mapsto (\hat{F}(X), 1)$ (Poincaré map on I) and time of first return $\hat{T}(X) > 0$ given explicitly by

$$\hat{T}(X) \;=\; \alpha + \log X^{-1/b} + \beta(X^\lambda)X^\lambda, \tag{3.3}$$

$$\hat{F}(X) \;=\; \gamma(X^\lambda)X^\lambda. \tag{3.4}$$

Set $\gamma(0) = \gamma_0$. Since $\lambda > 1$, it follows that we can assume that $A_0 > 0$ is chosen sufficiently small so that that $\hat{F}(X) < X/2$, for all $X \in (0, A_0]$. Noting the explicit form of the return map (3.4) we recover a well-known result.

Lemma 3.1. *The set Σ is a minimal attractor for ϕ_t whenever $\lambda > 1$.*

Given $X \in (0, A_0]$, define sequences (X_n), (γ_n) and $(T_n(X))$ by

$$X_0 = X, \quad X_n = \gamma(X_{n-1}^\lambda)X_{n-1}^\lambda, \; n \geq 1,$$

$$\gamma_n = \gamma(X_n^\lambda), \; n \geq 0, \quad T_n(X) = \hat{T}(X_{n-1}), \; n \geq 1.$$

It follows that

$$\begin{aligned} X_n &= \gamma_{n-1} X_{n-1}^{\lambda}, \\ &= (\Pi_{j=0}^{n-1} \gamma_{n-1-j}^{\lambda^j}) X^{\lambda^n}, \\ &= \bar{\gamma}_n X^{\lambda^n}, \quad n \geq 1, \end{aligned}$$

where $\bar{\gamma}_n(X) = \Pi_{j=0}^{n-1} \gamma_{n-1-j}^{\lambda^j}$, $n \geq 1$, and we define $\bar{\gamma}_0(X) \equiv 1$. It follows from (3.3) that for $X \in (0, A_0]$ we have

$$T_n(X) = \alpha - \frac{1}{b} \log(\bar{\gamma}_{n-1}(X)) + \lambda^{n-1} \log X^{-\frac{1}{b}} + \beta(\bar{\gamma}_{n-1}(X)^{\lambda} X^{\lambda^n}) \bar{\gamma}_{n-1}(X)^{\lambda} X^{\lambda^n}. \tag{3.5}$$

For our subsequent results, we need some estimates on the functions $\bar{\gamma}_n(X)$ and their derivatives.

Lemma 3.2. *Let $\tau > \gamma_0 > 0$; then there exists $A_1 = A_1(\tau) \in (0, A_0]$ such that for $X \in (0, A_1]$ and $n \geq 0$ we have*

$$\bar{\gamma}_n(X) \leq \tau^{\frac{\lambda^n - 1}{\lambda - 1}}.$$

Proof. We start by constructing A_1. Since γ is C^1, we may choose $C > 0$ such that for all $Y \in (0, A_0]$

$$|\gamma(Y) - \gamma_0| \leq CY. \tag{3.6}$$

We define $A_1 \in (0, A_0]$ by requiring that for all $n \geq 1$

$$C\tau^{\frac{\lambda^n - \lambda}{\lambda - 1}} A_1^{\lambda^n} \leq C A_1^{\lambda} \leq \min\{A_0, \tau - \gamma_0\}. \tag{3.7}$$

Our proof now proceeds by induction on n. Since $\bar{\gamma}_0(X) \equiv 1$, the result is trivially true for $n = 0$. Suppose we have verified the estimate for $n - 1$. For $n \geq 1$ we have the following formula relating $\bar{\gamma}_n$ and $\bar{\gamma}_{n-1}$:

$$\bar{\gamma}_n(X) = \gamma(\bar{\gamma}_{n-1}(X)^{\lambda} X^{\lambda^n}) \bar{\gamma}_{n-1}(X)^{\lambda}. \tag{3.8}$$

It follows from our inductive hypothesis that $\bar{\gamma}_n(X) \leq \gamma(\bar{\gamma}_{n-1}(X)^{\lambda} X^{\lambda^n}) \tau^{\frac{\lambda^n - \lambda}{\lambda - 1}}$. Since by (3.7) $\bar{\gamma}_{n-1}(X)^{\lambda} X^{\lambda^n} \in (0, A_0]$, it follows from (3.6) that

$$|\gamma(\bar{\gamma}_{n-1}(X)^{\lambda} X^{\lambda^n}) - \gamma_0| \leq C\bar{\gamma}_{n-1}(X)^{\lambda} X^{\lambda^n} \leq C\tau^{\frac{\lambda^n - \lambda}{\lambda - 1}} X^{\lambda^n}.$$

Substituting in our estimate for $\bar{\gamma}_n(X)$, we have for $X \in (0, A_1]$

$$\begin{aligned}
\bar{\gamma}_n(X) &\leq \tau^{\frac{\lambda^n-\lambda}{\lambda-1}}(\gamma_0 + C\tau^{\frac{\lambda^n-\lambda}{\lambda-1}}X^{\lambda^n}), \\
&= \tau^{\frac{\lambda^n-1}{\lambda-1}}(\gamma_0 + C\tau^{\frac{\lambda^n-\lambda}{\lambda-1}}X^{\lambda^n})/\tau, \\
&\leq \tau^{\frac{\lambda^n-1}{\lambda-1}}(\gamma_0 + CA_1^{\lambda})/\tau, \\
&\leq \tau^{\frac{\lambda^n-1}{\lambda-1}},
\end{aligned}$$

where the last two inequalities follow from (3.7). □

Remark 3.3. It follows from Lemma 3.2 that if $\mu > 0$ and $0 < A\gamma_0^{\frac{\mu}{\lambda-1}} < 1$, then $X^{\lambda^n}\bar{\gamma}_n(X)^{\mu} \to 0$ very rapidly as $n \to \infty$, $X \in (0, A]$. We use this observation frequently in the proofs of the next three technical lemmas.

Lemma 3.4. *Let $C > 1$; then one can find $A_2 \in (0, A_1]$ such that for all $X \in (0, A_2]$ and $n \geq 1$ we have $|\bar{\gamma}_n'(X)| \leq C\lambda^n\bar{\gamma}_n(X)$.*

Proof. Fix $C > 1$ and define $\epsilon_n = \frac{c}{n^2}$, where $c > 0$ is chosen so that

$$\prod_{n=1}^{\infty}\left(1 + \frac{1}{\lambda}\frac{c}{n^2}\right) < C. \tag{3.9}$$

We show that we can choose $A_2 \in (0, A_1]$ such that if $X \in (0, A_2]$ then for all $n \geq 1$

$$|\bar{\gamma}_n'(X)| \leq \left(\Pi_{j=1}^n(\lambda + \epsilon_j)\right)\bar{\gamma}_n(X).$$

This suffices since $\Pi_{j=1}^n(\lambda + \epsilon_j) = \lambda^n\Pi_{j=1}^n(1 + \frac{1}{\lambda}\frac{c}{j^2})$, and so it follows from (3.9) that for all $n \geq 1$ we have $\Pi_{j=1}^n(\lambda + \epsilon_j) \leq C\lambda^n$.

Our proof proceeds by induction on n. If $n = 1$, $\bar{\gamma}_1(X) = \gamma(X^{\lambda})$ and so $\bar{\gamma}_1'(X) = \lambda\gamma'(X^{\lambda})X^{\lambda-1}$. Hence

$$\frac{\bar{\gamma}_1'(X)}{\bar{\gamma}_1(X)} = \lambda X^{\lambda-1}\frac{\gamma'(X^{\lambda})}{\gamma(X^{\lambda})}.$$

Since γ is C^1, it follows that we may choose $A_2 \in (0, A_1]$ so that for all $X \in (0, A_2]$ we have $|\lambda X^{\lambda-1}\frac{\gamma'(X^{\lambda})}{\gamma(X^{\lambda})}| \leq \lambda + \epsilon_1$. This establishes the first step of the induction.

Differentiating (3.8) with respect to X, we may write $\bar{\gamma}_n'(X) = I_1 + I_2 + I_3$ where

$$\begin{aligned}
I_1 &= \lambda\gamma(\bar{\gamma}_{n-1}(X)^{\lambda}X^{\lambda^n})\bar{\gamma}_{n-1}(X)^{\lambda-1}\bar{\gamma}_{n-1}'(X), \\
&= \lambda\bar{\gamma}_n(X)\frac{\bar{\gamma}_{n-1}'(X)}{\bar{\gamma}_{n-1}(X)}, \\
I_2 &= \lambda\gamma'(\bar{\gamma}_{n-1}(X)^{\lambda}X^{\lambda^n})\bar{\gamma}_{n-1}(X)^{2\lambda-1}\bar{\gamma}_{n-1}'(X)X^{\lambda^n}, \\
I_3 &= \lambda^n\gamma'(\bar{\gamma}_{n-1}(X)^{\lambda}X^{\lambda^n})\bar{\gamma}_{n-1}(X)^{2\lambda}X^{\lambda^n-1}.
\end{aligned}$$

Suppose we have verified the required estimate for $n-1$. It follows from the second expression for I_1 that

$$|I_1| \leq \lambda(\Pi_{j=1}^{n-1}(\lambda+\epsilon_j))\bar{\gamma}_n(X). \tag{3.10}$$

Note that (3.10) holds without further conditions being required on X.

Rearranging our expression for I_2, we find that

$$\begin{aligned} I_2 &= \gamma(\bar{\gamma}_{n-1}(X)^\lambda X^{\lambda^n})\bar{\gamma}_{n-1}(X)^\lambda \frac{\bar{\gamma}'_{n-1}(X)}{\bar{\gamma}_{n-1}(X)}\lambda X^{\lambda^n}\bar{\gamma}_{n-1}(X)^\lambda \frac{\gamma'(\bar{\gamma}_{n-1}(X)^\lambda X^{\lambda^n})}{\gamma(\bar{\gamma}_{n-1}(X)^\lambda X^{\lambda^n})}, \\ &= \bar{\gamma}_n(X)\left(\frac{\bar{\gamma}'_{n-1}(X)}{\bar{\gamma}_{n-1}(X)}\right)\lambda X^{\lambda^n}\bar{\gamma}_{n-1}(X)^\lambda\left(\frac{\gamma'(\bar{\gamma}_{n-1}(X)^\lambda X^{\lambda^n})}{\gamma(\bar{\gamma}_{n-1}(X)^\lambda X^{\lambda^n})}\right). \end{aligned}$$

Choosing A_2 smaller if necessary (we may do this uniformly in n using lemma 3.2 and remark 3.3), we may require that

$$\left|\lambda X^{\lambda^n}\bar{\gamma}_{n-1}(X)^\lambda \frac{\gamma'(\bar{\gamma}_{n-1}(X)^\lambda X^{\lambda^n})}{\gamma(\bar{\gamma}_{n-1}(X)^\lambda X^{\lambda^n})}\right| \leq \frac{\epsilon_n}{2},$$

for all $n \geq 2$ and $X \in (0, A_2]$. Hence we obtain the following estimate on $|I_2|$

$$|I_2| \leq \frac{\epsilon_n}{2}(\Pi_{j=1}^{n-1}(\lambda+\epsilon_j))\bar{\gamma}_n(X). \tag{3.11}$$

Rearranging our expression for I_3, we find that

$$I_3 = \bar{\gamma}_n(X)\frac{\gamma'(\bar{\gamma}_{n-1}(X)^\lambda X^{\lambda^n})}{\gamma(\bar{\gamma}_{n-1}(X)^\lambda X^{\lambda^n})}\bar{\gamma}_{n-1}(X)^\lambda X^{\lambda^n-1}\lambda^n$$

Choosing A_2 smaller if necessary (again using lemma 3.2, remark 3.3 as in the estimate of I_2), we may require that

$$\left|\frac{\gamma'(\bar{\gamma}_{n-1}(X)^\lambda X^{\lambda^n})}{\gamma(\bar{\gamma}_{n-1}(X)^\lambda X^{\lambda^n})}\bar{\gamma}_{n-1}(X)^\lambda X^{\lambda^n-1}\lambda^n\right| \leq \frac{\epsilon_n}{2}\Pi_{j=1}^{n-1}(\lambda+\epsilon_j),$$

for all $X \in (0, A_2]$, $n \geq 2$. Hence we obtain the following estimate on $|I_3|$:

$$|I_3| \leq \frac{\epsilon_n}{2}(\Pi_{j=1}^{n-1}(\lambda+\epsilon_j))\bar{\gamma}_n(X). \tag{3.12}$$

Estimates (3.10, 3.11, 3.12) together complete the inductive step. □

Let $S_n(X) = S_n = \sum_{j=1}^n T_j(X)$. It follows from (3.5) that

$$S_n = n\alpha - \frac{1}{b}\log\Pi_{j=1}^{n-1}\bar{\gamma}_j(X) + \frac{\lambda^n-1}{\lambda-1}\log X^{-\frac{1}{b}} + \sum_{j=1}^{n}\beta(\bar{\gamma}_{j-1}(X)^\lambda X^{\lambda^j})\bar{\gamma}_{j-1}(X)^\lambda X^{\lambda^j} \tag{3.13}$$

This formula for the return times of a homoclinic attractor in $\mathbf{R}^2$ is completely general.

Next we need estimates on the derivative of S_n.

Lemma 3.5. *We may choose $A_3 \in (0, A_2]$, such that for all $n \geq 1$, $X \in (0, A_3]$ we have $S'_n(X) = M_1(X) + M_2(X) + M_3(X)$, where*

$$\text{(M1)}\ \ |M_1(X)| \leq \frac{2}{b}\frac{\lambda^n - \lambda}{\lambda - 1},\ \ \text{(M2)}\ \ M_2(X) = -\frac{1}{b}\frac{\lambda^n - 1}{\lambda - 1}\frac{1}{X},\ \ \text{(M3)}\ \ |M_3(X)| \leq \frac{2}{b}.$$

Proof. Differentiating S_n with respect to X, we may write $S'_n(X) = M_1(X) + M_2(X) + M_3(X)$ where

$$\begin{aligned} M_1(X) &= -\frac{1}{b}\sum_{j=1}^{n-1}\frac{\bar\gamma'_j(X)}{\bar\gamma_j(X)}, \\ M_2(X) &= -\frac{1}{b}\frac{\lambda^n - 1}{\lambda - 1}\frac{1}{X}, \\ M_3(X) &= \sum_{j=1}^{n}(a_j(X) + b_j(X) + c_j(X)), \end{aligned}$$

and the terms a_j, b_j, c_j are given explicitly by

$$\begin{aligned} a_j(X) = \beta'(\bar\gamma_{j-1}(X)^\lambda X^{\lambda^j})\bar\gamma_{j-1}(X)^\lambda X^{\lambda^j}[&\lambda\bar\gamma_{j-1}(X)^{\lambda-1}\bar\gamma'_{j-1}(X)X^{\lambda^j} \\ &+ \bar\gamma_{j-1}(X)^\lambda\lambda^j X^{\lambda^j - 1}], \end{aligned} \tag{3.14}$$

$$b_j(X) = \beta(\bar\gamma_{j-1}(X)^\lambda X^{\lambda^j})\lambda\bar\gamma_{j-1}(X)^{\lambda-1}\bar\gamma'_{j-1}(X)X^{\lambda^j}, \tag{3.15}$$

$$c_j(X) = \beta(\bar\gamma_{j-1}(X)^\lambda X^{\lambda^j})\bar\gamma_{j-1}(X)^\lambda\lambda^j X^{\lambda^j - 1}. \tag{3.16}$$

Choose $A_3 \in (0, A_2]$ so that the estimate of lemma 3.4 applies with $C = 2$. Estimating the sum for $M_1(X)$ using the estimate of lemma 3.4 yields estimate (M1). In order to estimate $M_3(X)$ it suffices to show that we can choose A_3 so that $|a_j(X)|, |b_j(X)|,\ |c_j(X)| \leq \frac{1}{3}2^{-j}$ for all j and $X \in (0, A_3]$. This is a routine estimate using lemma 3.2, remark 3.3 and we omit details. □

For $j \geq 1$, define C^1 functions on $[0, A_3]$ by

$$\begin{aligned} p_j(X) &= \frac{\bar\gamma'_j(X)}{\bar\gamma_j(X)}, \\ m_j(X) &= a_j(X) + b_j(X) + c_j(X), \end{aligned}$$

where a_j, b_j, c_j are defined according to (3.14,3.15,3.16).

Lemma 3.6. *We may choose $C > 0$, $A_4 \in (0, A_3]$ such that for all $n \geq 1$, $X \in (0, A_4]$ we have*

$$\begin{aligned} |p_n'(X)| &\leq C\lambda^n, \\ |m_n'(X)| &\leq C2^{-n}. \end{aligned}$$

Proof. For $j \geq 0$, define

$$q_j(X) = \frac{\gamma'(\bar{\gamma}_j(X)^\lambda X^{\lambda^j})}{\gamma(\bar{\gamma}_j(X)^\lambda X^{\lambda^j})}.$$

Straightforward estimation, using Lemma 3.4, shows that we can choose $c > 0$ such that for all $X \in [0, A_3]$, $j \geq 0$,

$$|q_j(X)| \leq c, \;\; |q_j'(X)| \leq c\lambda^j \bar{\gamma}_j(X)^\lambda X^{\lambda^j}. \tag{3.17}$$

Just as in the proof of Lemma 3.4, it follows from (3.8) that

$$\begin{aligned} p_n(X) &= \lambda p_{n-1}(X)(1 + q_{n-1}(X)\bar{\gamma}_{n-1}(X)^\lambda X^{\lambda^n}) \\ &\quad + \lambda^n q_{n-1}(X)\gamma_{n-1}(X)^\lambda X^{\lambda^n - 1}. \end{aligned}$$

Differentiating this equation, we obtain an equation relating $p_n'(X)$ to $p_{n-1}'(X)$, $q_{n-1}'(X)$ and derivatives of $\bar{\gamma}_{n-1}(X)^\lambda X^{\lambda^n}$ and $\gamma_{n-1}(X)^\lambda X^{\lambda^n - 1}$. Using remark 3.3, we may suppose $A_4 \in (0, A_3]$ chosen so that all these terms, with the exception of the derivatives p_n', p_{n-1}', go to zero very rapidly as $j \to \infty$. It follows, just as in the proof of Lemma 3.4, that we can find $C > 0$ so that p_n' satisfies the required estimate, all $n \geq 1$.

The proof of the estimate on the m_j' is straightforward and omitted. □

Corollary 3.7. *We may choose $d > 0$, $A_5 \in (0, A_4]$ and $N_0 \geq 1$ such that if $n > m \geq N_0$, then*

1. $|S_n'(X) - S_m'(X)| \geq d\lambda^n \frac{1}{X}$, $X \in (0, A_5]$.
2. $S_n' - S_m'$ *is monotonic on* $(0, A_5]$.

Proof. Since $\lambda > 1$, (1) follows easily from lemma 3.5. Using the estimates of Lemma 3.6, together with lemma 3.5, we easily show that we can choose $A_5 \in (0, A_4]$ so that for all $X \in (0, A_5]$, $n > m$, $S_n''(X) - S_m''(X)$ is strictly positive. □

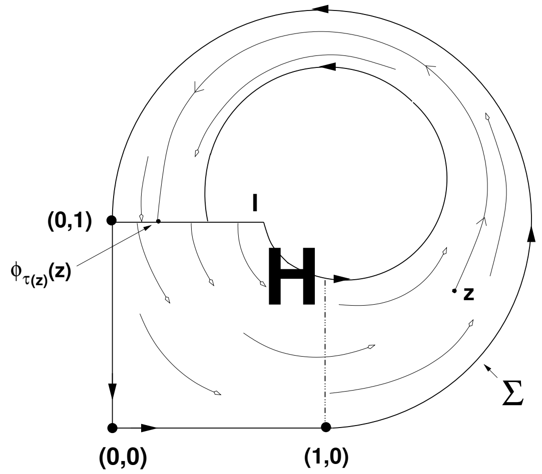

Figure 3. The region H and the map τ

Let $H = \cup_{X \in I} \phi_{[0,T(X)]}(X)$ denote the compact region in $\mathbf{R}^2$ swept out by the (semi)flow ϕ_t (see figure 3). Let $\xi : \mathbf{R} \to \Sigma$ be the trajectory of ϕ_t satisfying $\xi(0) = (0,1)$. Thus, $\lim_{t\to\pm\infty} \xi(t) = (0,0)$.

Let $z \in H \setminus \{(0,0)\}$. We define $\tau(z) \in \mathbf{R}(\geq 0)$ to be the smallest value of $t \geq 0$ such that $\phi_{\tau(z)}(z) \in I$. The function τ is smooth on $H \setminus I$ and vanishes identically on I. It follows from the continuity of ϕ_t that $\xi(-\tau(z)) \to x$ as $z \to x \in \Sigma$, $x \neq 0$.

While $\tau(z)$ defines the first time the trajectory through z meets I, subsequent return times are given in terms of $S_n(\phi_{\tau(z)}(z))$. Specifically, the nth return time for the trajectory through z to I is equal to $\tau(z) + S_n(\phi_{\tau(z)}(z))$.

Lemma 3.8. *Let $T \in \mathbf{R}$. For any $z \in H \setminus \Sigma$ and divergent monotone increasing sequence (t_n) the following are equivalent:*

(i) $\lim_{n\to\infty} \phi_{t_n}(z) = \xi(T)$.

(ii) There is an increasing sequence of integers k_n such that

$$\lim_{n\to\infty} t_n - S_{k_n}(\phi_{\tau(z)}(z)) - \tau(z) = T.$$

In the case $T = \pm\infty$, (i) $\Longrightarrow$ (ii) but, in general, (ii) $\not\Longrightarrow$ (i).

Proof. In order to simplify the notation, we set $\tau = \tau(z)$, $u_1 = \phi_\tau(z)$, $S_n = S_n(u_1)$, and $u_n = \phi_{S_n}(u_1) \in I$. We remark that $\lim_{n\to\infty} u_n = \{(0,1)\}$. We first show that

(ii) implies (i). We have $\phi_{S_{k_n}+\tau+T}(z) = \phi_{S_{k_n}+T}(u_1) = \phi_T(u_n)$, and so

$$\lim_{n\to\infty} \phi_{S_{k_n}+\tau+T}(z) = \lim_{n\to\infty} \phi_T(u_n) = \phi_T\left((0,1)\right) = \xi(T).$$

Conversely, suppose that $\phi_{t_n}(z) \to \xi(T)$. Since $\phi_{t_n}(z) = \phi_{t_n-\tau}(u_1)$ it follows that

$$\phi_{t_n-\tau-T}(u_1) \to \{(0,1)\}.$$

Hence, since $\phi_T(z)$ is continuous in z, it follows that $t_n - \tau - T - S_{k_n} \to 0$ as $n \to \infty$ for some sequence $k_n \to \infty$.

Finally, exactly the same arguments work to show that (i) $\Longrightarrow$ (ii) in case $T = \pm\infty$. On the other hand the converse does not hold; if $T = +\infty$ we may choose a very rapidly increasing sequence (t_n) satisfying $\lim_{n\to\infty} \phi_{t_n}(z) = \xi(0)$ and then choose a slowly increasing sequence (k_n) so as to satisfy (ii) with $T = \infty$. A similar argument covers the case $T = -\infty$. □

4. Product of a cycle and a periodic orbit

In this section we consider the product of a periodic orbit and the homoclinic attractor $\Sigma \subset H \subset \mathbf{R}^2$ constructed in the previous section. The (semi)flow on $(z,\theta) \in H \times S^1$ is given by $\Phi_t(z,\theta) = (\phi_t(z), \theta + \varpi t \pmod 1)$, with $\phi_t : H \to H$ the (semi)flow defined in the previous section. Note that ϖ is the frequency of the periodic orbit and the cycle has slowing down ratio $\lambda > 1$.

Lemma 4.1. *For all $\varpi \neq 0$, $\theta \in \mathbf{R}$, $\lambda > 1$, and almost all $u \in I \setminus \{(0,1)\}$, $\omega(u,\theta) = \Sigma \times S^1$.*

Proof. Let $u \in I \setminus \{(0,1)\}$. We consider the set of all intersections of forward trajectories through (u,θ) with I by defining $u_n \in I$, $t_n \in \mathbf{R}(>0)$, and $\theta_n \in S^1$ so that $\Phi_{t_n}(u,\theta) = (u_n,\theta_n)$.

We have $\omega(u,\theta) = \Sigma \times S^1$ if and only if (θ_n) is dense in $S^1 = [0,1)$. By lemma 3.8 we see that $t_n = S_n(u)$ and $\theta_n = \theta + \varpi S_n(u)$. Hence we have density in $[0,1)$ if $[\theta + \varpi S_n(u)]$ is uniformly distributed in $[0,1)$ ($[r]$ denotes the fractional part of r). It follows by Corollary 3.7 that for $n > m \geq N_0$, $S_n'(u) - S_m'(u)$ is monotonic on $(0, A_4]$ and is bounded below by a multiple of λ^n. It follows from theorem 5.10, corollary 2 of Harman (1998) that $[\theta + \varpi S_n(u)]$ is uniformly distributed in $[0,1)$ for almost all $u \in (0, A_4]$. □

Proof of Theorem 1.2 Since u_1 depends smoothly and invertibly on z for a full measure set of $z \in H$, we may apply lemma 4.1. It follows that for all ϖ, $\theta \in S^1$ and $\lambda > 1$, $\omega(z,\theta) = \Sigma \times S^1$ for almost all $z \in H$. Hence $\Sigma \times S^1$ is a minimal attractor. □

Remarks 4.2. (1) Note that it does not follow from the proof of theorem 1.2 that $\omega(z,\theta) = \Sigma \times S^1$ for all $z \in H$. Indeed, this is generally false. As a simple example, suppose $\lambda > 1$ is an integer, $\hat{\gamma}(0) = 1$, and $S_n(u) = \log(u_1^{-1/b})\lambda^n$. If we choose $u \in I, \omega$ such that $\log(u_1^{-1/b}) = p/q$ and $\varpi = s/q$, then ϖS_n is clearly rational with denominator bounded by $q(\lambda - 1)$.
(2) Our proof of theorem 1.2 assumes the dynamics on the second factor is time-periodic. The proof obviously extends to the case when the dynamics on the second factor is given by a semiflow defined on a neighbourhood of a *hyperbolic* attracting limit cycle.
(3) As one of the referees pointed out to us, an approach based on expanding sequences, see Melbourne & Stewart (1997), may be used to obtain partial results on the ω-limit sets of points in $H \times S^1$. In particular, for almost all frequencies ϖ, there exist trajectories in $H \times S^1$ with ω-limit set equal to $\Sigma \times S^1$. However, without more detailed estimates on the flow on H, this approach does not yield a proof that $\Sigma \times S^1$ is a (minimal) Milnor attractor, as the expanding sequence depends nonlinearly on the initial point $u \in I \setminus \{(0,1)\}$.

5. Product of an attracting cycle and a chaotic set

In this section, we consider a flow which is the product of a homoclinic attractor with a transitive hyperbolic flow $\psi_t : X \to X$. Under these assumptions on ψ_t, we can prove that the product of the cycle with the the hyperbolic flow is a Milnor attractor. However, as was pointed out to us by Ian Melbourne, our arguments can be substantially simplified if we assume in addition that the flow ψ_t is topologically mixing. Under this additional assumption, the product of the cycle with the the hyperbolic flow is a *minimal* Milnor attractor. Providing that we work in a sufficiently high smoothness class, topologically mixing hyperbolic flows are generically stably mixing (we refer to Field *et al.* (2003) for precise statements of results and background). For these reasons, we emphasize the case when ψ_t is topologically mixing and only give brief details on the methods needed when ψ_t is not mixing.

We start by briefly reviewing some definitions and results on hyperbolic flows (we refer to Bowen & Ruelle (1975) for further details). Recall that if $\psi_t : M \to M$ is a smooth flow on the Riemannian manifold M, then a compact ψ_t-invariant set $X \subset M$ is called a *basic set* for the flow if X is hyperbolic, transitive, locally maximal and isolated. Henceforth we regard ψ_t as a flow on the basic set X. The flow ψ_t is *topologically mixing* if for all non-empty open subsets U, V of X (induced topology), there exists $T = T(U, V)$ such that $\psi_t(U) \cap V \neq \emptyset$, $t \geq T$. We fix an equilibrium state ν on X corresponding to a Hölder continuous potential, for example the measure of maximal entropy. The measure ν is a regular Borel measure which is strictly positive on open subsets of X and the flow ϕ_t is ν-measure preserving and ergodic. If ψ_t is topologically mixing, then ψ_t is measure theoretically mixing in the sense that for all measurable subsets A, B of X, we have

$$\lim_{t\to\infty} \nu(\Phi_t(A) \cap B) = \nu(A)\nu(B).$$

It is well-known that for hyperbolic basic sets, topologically mixing is equivalent to measure theoretic mixing, provided that the measure is an equilibrium state corresponding to a Hölder continuous potential. In future, we often just say 'mixing'.

We consider the product of a homoclinic attractor $\Sigma \subset H$ with ψ_t. We prove in theorem 5.1 that the product $\Sigma \times X$ is a minimal Milnor attractor which is equal to the likely limit set of $H \times X$. For simplicity, we work with a homoclinic attractor that satisfies the conditions† of §3. However, we expect that much of what we say generalizes fairly straightforwardly to general heteroclinic attractors in $\mathbf{R}^2$ or $\mathbf{R}^n$.

Specifically, we assume that $\Sigma \subset H \subset \mathbf{R}^2$ is a one dimensional homoclinic attractor for the semiflow $\phi_t : H \to H$ defined on a connected subset H of $\mathbf{R}$ contained in $\mathbf{R}^2$. We suppose that H is a one-sided open neighbourhood of Σ chosen so that if $z \in H \setminus \Sigma$, then $\omega(z) = \Sigma$. We assume the component ∂H of the boundary of H disjoint from Σ is smooth and transverse to the flow ϕ_t. Denote the hyperbolic saddle point on Σ by q and define $\lambda = a/b > 1$, where $-a < 0 < b$ are the eigenvalues of the Jacobian at q. Choose a section T to the cycle, transverse to

† We do not need to assume the nonresonance conditions $a/b \notin \{3/2, 2, 3, 4\}$ in this case.

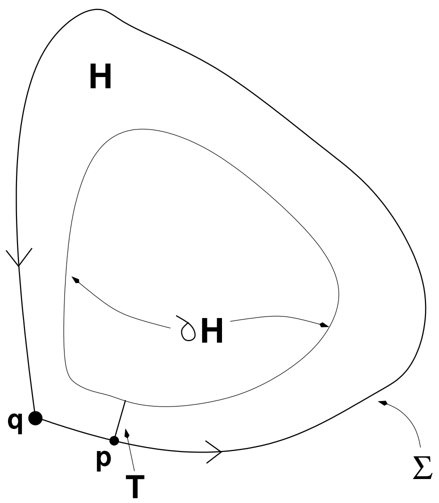

Figure 4. An attracting homoclinic cycle Σ with section T transverse to the cycle at P.

the connection in Σ, such that all trajectories in $H \setminus \Sigma$ pass through T and return after finite time – see figure 4. Set $\Sigma \cap T = \{p\}$.

We consider the flow $\Phi_t(u, x) = (\phi_t(u), \psi_t(x))$ on $H \times X$.

Theorem 5.1. *Suppose that ψ_t is topologically mixing. Then the product $\Sigma \times X$ is the likely limit set of $H \times X$ and is a minimal Milnor attractor for Φ_t*

Proof. Let $Z = \partial H \times X$. We take the product measure $\mu = \ell \times \nu$ on Z where ℓ is Lebesgue measure on ∂H and ν is an equilibrium state on X. Let $\epsilon > 0$. Given $x \in X$, let $D_\epsilon(x) \subset X$ denote the open ball radius ϵ, centre x where distance is induced from the metric on the ambient manifold M. Set $B_\epsilon(x) = \{p\} \times D_\epsilon(x)$. It follows from Lemma 2.3(2) that it suffices to show that if $K \subset Z$ is any measurable set with $\mu(K) > 0$, then $\mu(\{a \in K \mid B_\epsilon(x) \cap \omega(a) \neq \emptyset\}) > 0$, for all $x \in X$ (note that we use Fubini's theorem and the smoothness of Φ_t to reduce to measurable subsets K of Z rather than $H \times X$). Suppose then that $K \subset Z$ is measurable and $\mu(K) > 0$. For each $\sigma \in \partial H$, let $K_\sigma = K \cap (\{\sigma\} \times X)$. It follows from Fubini's theorem that we can choose a measurable subset J of ∂H such that for all $\sigma \in J$, K_σ is ν-measurable, $\nu(K_\sigma) > 0$ and $\int_J \nu(K_\sigma)\, d\ell = \mu(K)$. Given $\sigma \in J$, let (t_n) be the corresponding sequence of return times to the section T. That is, $\phi_{t_n}(\sigma) \in T$, $n = 1, 2, \ldots$. Define $E^\sigma_n = \psi_{-t_n}(D_\epsilon(x)) \cap K_\sigma$. Then

$$\begin{aligned}\limsup E_n^\sigma &= \{x \in K_\sigma \mid \forall m \ \ \exists n \geq m \ \text{ such that } x \in E_n^\sigma\} \\ &= \{x \in K_\sigma \mid \psi_{t_n}(x) \in D_\epsilon(x) \ \text{ infinitely often}\}.\end{aligned}$$

It follows by Fatou's lemma that

$$\begin{aligned}\nu(\limsup E_n^\sigma) &\geq \limsup \nu(E_n^\sigma) &(5.1)\\ &= \nu(K_\sigma)\nu(D_\epsilon(x)), &(5.2)\end{aligned}$$

since ψ_t is mixing. Define $\limsup E_n = \cup_{\sigma\in J} \limsup E_n^\sigma$. Then $\limsup E_n$ is a measurable subset of Z and, by Fubini's theorem and (5.2), we have $\mu(\limsup E_n) \geq \mu(K)\nu(D_\epsilon(x)) > 0$. Hence, $\mu(\{a \in K \mid B_\epsilon(x) \cap \omega(a) \neq \emptyset\}) > 0$. □

(*a*) *The case when* ψ_t *is not mixing*

We have the following result that covers the case when ψ_t is not mixing.

Theorem 5.2. *Suppose that* ψ_t *is transitive. Then the product* $\Sigma \times X$ *is the likely limit set of* $H \times X$ *and is a Milnor attractor for* Φ_t

Proof. We only give brief details of the proof. We start by proving the result in case $\psi_t : \mathcal{S}^r \to \mathcal{S}^r$ is the suspension of a subshift of finite type $\mathcal{S}$ with roof function r. This is proved using fairly standard methods based on symbolic coding in combination with estimates on the return times to the section T. Next suppose X is a basic set. Following Bowen & Ruelle (1975), let $\pi : \mathcal{S}^r \to X$ define a symbolic dynamics on X and let $\Pi = I_\Sigma \times \pi : \Sigma \times \mathcal{S}^r \to \Sigma \times X$ denote the associated finite-to-one projection. Choose an equilibrium state m on $\mathcal{S}^r$ and corresponding measure m^X on X (so that π is a measure preserving isometry). Noting that $\omega(\Pi(z, s)) = \Pi(\omega(z, s))$, $(z, s) \in \Sigma \times \mathcal{S}^r$, we see that the likely limit set for $H \times X$ is equal to the projection by Π of the likely limit set for $H \times \mathcal{S}^r$. □

(*b*) *Product of a homoclinic attractor and a hyperbolic attractor*

Suppose that ψ_t is a flow on the manifold M and that X is a hyperbolic attractor for ψ_t. Let $\Phi_t(u, (x, s)) = (\phi_t(u), \psi_t(x, s))$ denote the corresponding semi-flow on $H \times M$.

Proof of Theorem 1.3 Take the Sinai-Ruelle-Bowen measure on X and apply theorems 5.1, 5.2. □

6. A product of attracting cycles

In this section we investigate dynamics for a product of two homoclinic attractors. In order to illuminate ideas rather than technicalities, we shall eventually make strong assumptions about the homoclinic attractors. However, we start with a result that holds for a quite general product dynamical system Φ_t on $M = H_1 \times H_2 \supset \Sigma_1 \times \Sigma_2$ defined by

$$\Phi_t(x_1, x_2) = (\phi_t^1(x_1), \phi_t^2(x_2)), \tag{6.1}$$

for semi-flows $\phi_t^k : H_k \to H_k$, $k = 1, 2$. For each $k = 1, 2$ we assume that ϕ_t^k has a single attractor, namely a homoclinic attractor $\Sigma_k \subset H_k$ with a fixed point q_k. We define sections T_k to $\Sigma_k \subset H_k$ meeting Σ_k at $p_k \neq q_k$, $k = 1, 2$. Set $T_k^\star = T_k \setminus \{p_k\}$ and $H_k^\star = H_k \setminus \Sigma_k$.

We define the invariant subset $\Sigma \subset \Sigma_1 \times \Sigma_2$ by

$$\Sigma = (\Sigma_1 \times \{q_2\}) \cup (\{q_1\} \times \Sigma_2). \tag{6.2}$$

This is a heteroclinic network that is contained within $\Sigma_1 \times \Sigma_2$ where only connections where one factor is at equilibrium are included.

Theorem 6.1. *The likely limit Λ for $H_1 \times H_2$ for the system (6.1) is either*

(a) $\Lambda = \Sigma$ as defined in (6.2) or

(b) $\Lambda = \Sigma_1 \times \Sigma_2$.

Proof. Obviously, $\omega(a, b) \subset \Sigma_1 \times \Sigma_2$ for all $(a, b) \in H_1 \times H_2$. Moreover there are no (a, b) in the full measure set $H_1^* \times H_2^*$ such that $\omega(a, b) \subset \{q_1\} \times \Sigma_2$ or $\omega(a, b) \subset \Sigma_1 \times \{q_2\}$. Hence $\Lambda \subset \Sigma_1 \times \Sigma_2$, $\Lambda \neq \{q_1\} \times \Sigma_2, \Sigma_1 \times \{q_2\}$.

Now suppose that Λ contains a point $(x_1, x_2) \in \Sigma_1 \times \Sigma_2 \setminus \Sigma$, such that $x_k \neq q_k$ for $k = 1, 2$. By theorem 1.1, Λ is a closed set that must contain $(\phi_t^1(x_1), \phi_s^2(x_2))$ for all $(t, s) \in \mathbf{R}^2$. Hence it must contain $\Sigma_1 \times \Sigma_2$ and we have case (b). The only remaining possibility is case (a). □

Proof of Theorem 1.4 This follows from theorem 6.1 on noting that the likely limit set of $H_1 \times H_2$ is a maximal Milnor attractor that contains all Milnor attractors of (6.1). □

Although it may seem counterintuitive, the examples we consider in the remainder of this section lead us to believe that the typical case is in fact (a). More precisely, we make the following conjecture:

Conjecture on products of attracting heteroclinic cycles: *Suppose that* Σ_1, Σ_2 *are attracting heteroclinic cycles consisting of the union of a finite set of trajectories and sets of equilibria* $\mathcal{E}_1$, $\mathcal{E}_2$. *For typical such pairs* (Σ_1, Σ_2) *we conjecture that the unique minimal Milnor attractor for the product system is equal to* $(\mathcal{E}_1 \times \Sigma_2) \cup (\Sigma_1 \times \mathcal{E}_2)$ *(that is, a union of a finite number of one-dimensional connections between the equilibria* $\mathcal{E}_1 \times \mathcal{E}_2$*).*

The remainder of this section is devoted to showing that the conjecture holds for a product of homoclinic attractors related to the model system described in §3.

(*a*) *Analysis of a model example*

We now assume that we are given a pair of homoclinic attractors, defined by planar flows as described in §3. In particular, each cycle Σ_k contains a hyperbolic saddle q_k, and has an associated asymptotic slowing-down ratio $\lambda_k > 1$, $k = 1, 2$.

We denote the connecting homoclinic orbit in Σ_k by $\xi_k(t)$. Thus, we have

$$\xi_k(t) = \phi_t^k(p_k) \in \Sigma_k \quad (k = 1, 2).$$

Let $S_n^k(u)$ denote the time to the nth return of $u \in T_k^\star$ to $T_k^\star$ for ϕ_t^k and $\tau_k(x)$ be the time of the first hit of the ϕ_t^k -orbit through $x \in H_k$ with T_k (see §3 for the precise definition of τ_k).

Suppose that the trajectory through $Z_2 = (u, z) \in T_1^\star \times H_2$ meets $T_1^\star \times H_2$ at the successive times $0 = t_0 < t_1 < \ldots$. The intersections define a sequence (u_n, z_n) with

$$\Phi_{t_n}(u, z) = (u_n, z_n) \in T_1^\star \times H_2. \tag{6.3}$$

We have $u_n \to p_1$ and $z_n \to \Sigma_2$ as $n \to \infty$. Let $\Omega_2(u, z) = \Omega_2(Z_2) = \mathcal{A}(\{z_n\}) \subset \Sigma_2$. We similarly define $\Omega_1(Z_1) \subset \Sigma_1$, given $Z_1 = (z', u') \in H_1 \times T_2^\star$. Observe that

$\omega(Z_k) = \Sigma_1 \times \Sigma_2$ if and only if $\Omega_k(Z_k) = \Sigma_k$, $k = 1, 2$. Of course, if $Z \in T_1^\star \times \Sigma_2$ (respectively, $\Sigma_1 \times T_2^\star$), then $\Omega_2(Z) = \Sigma_1 \times \{q_2\}$ (respectively, $\Omega_1(Z) = \{q_1\} \times \Sigma_2$).

We consider the case $(u, z) \in T_1^\star \times H_2$. If $\hat{z} \in \Omega_2(u, z)$, then either $\hat{z} = q_2$, in which case $\omega(u, z) \supset \Sigma_1 \times \{q_2\}$, or not. If $\hat{z} \neq q_2$, then there exists a unique $T \in \mathbf{R}$ such that $\hat{z} = \xi_2(T)$. Choose an increasing sequence (k_n) of integers such that $z_{k_n} \to \hat{z}$. Set $\tau_2(z) = v \in T_2^\star$ and note that it follows from lemma 3.8 that there is an increasing sequence (l_n) of integers such that

$$\lim_{n\to\infty} (t_{k_n} - S^2_{l_n}(v)) - \tau_2(z) = T.$$

It follows from (6.3) that

$$\lim_{n\to\infty} (S^1_{k_n}(u) - S^2_{l_n}(v)) - \tau_2(z) = T.$$

For $m, n \in \mathbf{N}$, let $\Theta_{m,n} = S^1_n(u) - S^2_m(v) - \tau_2(z)$. In order that $\hat{z} = \xi_2(T) \in \Omega_2(u, z)$, it is necessary and sufficient that $T \in \mathcal{A}(\{\Theta_{m,n} \mid m, n \in \mathbf{N}\})$. In particular, $\Omega_2(u, z) = \Sigma_2$ if and only if

$$\mathcal{A}(\{\Theta_{m,n} \mid m, n \in \mathbf{N}\}) = \mathbf{R}.$$

Fix $Z \in M$ and note that we have already covered the case where one or other component of Z lies in one of the factor cycles. Hence it is no loss of generality to assume $Z \in H_1^\star \times H_2^\star$. We consider the first intersection of the trajectory through Z with the $T_1^\star \times H_2^\star$ and $H_1^\star \times T_2^\star$ to be given by $Z_2 \in T_1^\star \times H_2^\star$ and $Z_1 \in H_1^\star \times T_2^\star$. Associated to the Z_k, $k = 1, 2$ we define sets $\{\Theta^k_{m,n} \mid m, n \in \mathbf{N}\}$ as described above.

Example 6.2. Suppose that the two homoclinic attractors (Σ_k, H_k) are defined by identical dynamics. Let $Z = (u, u) \in T_1^\star \times T_2^\star$. It is obvious that $\omega(Z)$ is the diagonal $\{(u, u) \mid u \in \Sigma_1\}$ of $\Sigma_1 \times \Sigma_2$. Moreover, the unique limit point $\hat{z} \in \Omega_2(Z)$ is equal to p_2 and so $T = 0$ and $\hat{z} = \xi_2(0)$. Note that the corresponding set $\{\Theta^2_{m,n} \mid m, n \in \mathbf{N}\}$ is *unbounded* and that it does *not* follow that q_2 lies in the limit set $\Omega_2(Z)$. Accumulation points of $\{\Theta^2_{m,n} \mid m, n \in \mathbf{N}\}$ give information about $\Omega_2(Z)$ but the unboundedness of $\{\Theta^2_{m,n} \mid m, n \in \mathbf{N}\}$ tells us nothing – indeed, $\{\Theta^2_{m,n} \mid m, n \in \mathbf{N}\}$ is always unbounded. However, if $\mathcal{A}(\{\Theta^2_{m,n} \mid m, n \in \mathbf{N}\}) = \emptyset$, then it follows that $q_2 \in \Omega_2(Z)$ and so $\omega(Z) \supset \Sigma_1 \times \{q_2\}$.

Generally, we have the following useful result.

Lemma 6.3. *Suppose that $Z \in H_1^\star \times H_2^\star$. If either $\mathcal{A}(\{\Theta^1_{m,n} \mid m,n \in \mathbf{N}\})$ or $\mathcal{A}(\{\Theta^2_{m,n} \mid m,n \in \mathbf{N}\})$ is empty then both are empty and $\omega(Z) = \Sigma$.*

It follows from our formulae for S_n^k (3.13) that there exist constants $C_k > 0$ such that $|S_n^k(u_k)| \sim C_k \lambda_k^n \log u_k^{-1}$.

For the remainder of this section we simplify our exposition by assuming that for our model cycles $\Sigma_k \subset H_k$, we can take $\alpha_k = 0$, $\beta_k \equiv 0$ and $\gamma_k \equiv 1$, $k = 1,2$. Under these assumptions†, it follows from (3.13) that

$$S_n^k(u_k) = \frac{\lambda_k^n}{b_k(\lambda_k - 1)} \log u_k^{-1} + \frac{1}{b_k(\lambda_k - 1)} \log u_k, \quad k = 1,2.$$

Choose $Z = (u,z) \in T_1^\star \times H_2$ as above. Then

$$\Theta_{m,n} = \Theta^2_{m,n} = \epsilon_1 \lambda_1^n - \epsilon_2 \lambda_2^m - \sigma \tag{6.4}$$

where ϵ_k and σ depend smoothly on the initial condition except on a set of zero measure:

$$\epsilon_1 = \frac{\log(u^{-1})}{b_1(\lambda_1 - 1)}, \quad \epsilon_2 = \frac{\log(v^{-1})}{b_2(\lambda_2 - 1)}, \tag{6.5}$$

$$\sigma = \frac{\log(u)}{b_1(\lambda_1 - 1)} - \frac{\log(v)}{b_2(\lambda_2 - 1)} + \tau_2(z,v). \tag{6.6}$$

It follows from our earlier discussion that a necessary and sufficient condition for $\hat{z} = \xi_2(T) \in \Omega_2(Z) \setminus \{q_2\}$ is that

$$\sigma + T \in \mathcal{A}(\{\epsilon_1 \lambda_1^n - \epsilon_2 \lambda_2^m \mid m,n \in \mathbf{N}\}). \tag{6.7}$$

(*b*) *Resonant eigenvalues*

Suppose that $\lambda_1 \neq \lambda_2 > 1$ are resonant. That is, $\lambda_1^r = \lambda_2^s$, for coprime $r,s \in \mathbf{N}$. Setting $\beta = \lambda_1^{\frac{1}{s}} = \lambda_2^{\frac{1}{r}} > 1$, we may write $\lambda_1 = \beta^s, \quad \lambda_2 = \beta^r$.

Lemma 6.4. *Let $Z \in H_1^\star \times H_2^\star$. Suppose that $Z = (u,z) \in T_1^\star \times H_2^\star$. If $\lambda_1, \lambda_2 > 1$ are resonant and $\epsilon_1 \neq \beta^j \epsilon_2$ all $j \in \mathbf{Z}$, then $\omega(Z) = \Sigma$. On the other hand if $\epsilon_1 = \beta^j \epsilon_2$ for some $j \in \mathbf{Z}$, then there is a point $\hat{Z} \in H_2^\star$ such that*

1. *If $r > 1, s = 1$, then $\omega(Z)$ is the union of $\Sigma_1 \times \{q_2\}$ with the Φ_t-trajectory through $(p_1, \hat{Z})$.*

† Which amount to the cycles and flows being obtained by identifying the edges I and J in figure 2.

2. *If* $r = 1, s > 1$, *then* $\omega(Z)$ *is the union of* $\{q_1\} \times \Sigma_2$ *with the* Φ_t*-trajectory through* $(p_1, \hat{Z})$.

3. *If* $r, s > 1$, *then* $\omega(Z)$ *is the union of* Σ *with the* Φ_t*-trajectory through* $(p_1, \hat{Z})$.

Proof. We have $\epsilon_1\lambda_1^n - \epsilon_2\lambda_2^m = \epsilon_1\beta^{ns} - \epsilon_2\beta^{mr}$. If $\epsilon_1 \neq \beta^j\epsilon_2$, all $j \in \mathbf{Z}$, then $\mathcal{A}(\{\Theta^2_{m,n} \mid m, n \in \mathbf{N}\}) = \emptyset$ and so the result follows from lemma 6.3.

Suppose, on the other hand, that for some $j \in \mathbf{Z}$, we have $\epsilon_1 = \beta^j\epsilon_2$. Since $(r, s) = 1$, we may find strictly positive integers n_0, m_0 such that $n_0 s - m_0 r = j$. Define the increasing sequence (n_k, m_k) by $n_k = n_0 + kr$, $m_k = m_0 + ks$ and note that $n_k s - m_k r = j$, $k \geq 1$. It follows that $\epsilon_1\beta^{n_k s} - \epsilon_2\beta^{m_k r} = 0$, $k \geq 1$ and so 0 is a limit point of (6.7). Hence $\sigma(u, z) + T = 0$ implicitly determines an additional limit point $\hat{Z} \in \Omega_2(u, z)$ and so $\omega(Z)$ contains the Φ_t-trajectory through $(p_1, \hat{Z})$. Examination of the possible sequences of return times shows that three possibilities may occur, as described in the statement of the theorem. □

(*c*) *Rapid approximation by rationals*

Definition 6.5. Suppose that α, β, γ are real numbers, with $\gamma > 1$ and $\alpha > 0$. We say that α is (β, γ)-*Liouville* if there are infinitely many positive integer pairs (p, q) such that

$$\left|\alpha - \frac{p}{q} - \frac{1}{q}\beta\right| \leq \frac{1}{q}\gamma^{-q}.$$

Remarks 6.6. (1) If $\eta > 1$, $K > 0$, and for infinitely many positive integer pairs (p, q) α satisfies the estimate $\left|\alpha - \frac{p}{q} - \frac{1}{q}\beta\right| \leq K\eta^{-q}$, then α is (β, γ)-Liouville for all $\gamma \in (1, \eta)$.
(2) A (β, γ)-Liouville number satisfies a *restricted Diophantine condition* (see Chapter 6, Harman 1998).
(3) Without loss of generality we can assume that $0 \leq \beta < 1$ since α is (β, γ)-Liouville if and only if it is $(\beta + n, \gamma)$-Liouville for any $n \in \mathbf{Z}$.

Examples 6.7. (1) The real number α is *Liouville* if for each n, we can choose an integer pair (p, q) such that $\left|\alpha - \frac{p}{q}\right| < \frac{1}{q^n}$. Normally, it is required that α is irrational. However, we regard rational numbers as (trivially) Liouville. If α is $(0, \gamma)$-Liouville, then α is Liouville. In particular, every rational number is $(0, \gamma)$-Liouville. Indeed, if α is $(0, \gamma)$-Liouville, then $\left|\alpha - \frac{p}{q}\right| \leq \frac{1}{q}\gamma^{-q}$ for infinitely many (p, q). Fix

$n \geq 1$ and choose (p, q) so that $\frac{1}{q}\gamma^{-q} < q^{-n}$. It follows that $|\alpha - \frac{p}{q}| < q^{-n}$ and so α is Liouville, but the converse is false. Most Liouville numbers are not $(0,\gamma)$-Liouville for any $\gamma > 1$. It is easy to show that there do exist irrational $(0,\gamma)$-Liouville numbers. For example, if we define $\gamma = \sum_{n=0}^{\infty} 10^{-p_n}$, where $p_0 = 1$ and $p_{n+1} = 10^{p_n}$, $n > 0$, then γ is $(0,\gamma)$-Liouville for all $\gamma \in (1, 10)$ (note Remark 6.6).
(2) For any rational β, if α is (β,γ)-Liouville then α is Liouville. This follows from the first remark, since if α is (β,γ)-Liouville and $\beta = r/s$, then α is $(0, \gamma^{\frac{1}{s}})$-Liouville. Conversely, if α is $(0,\gamma)$-Liouville then, for every rational β, α is (β,η)-Liouville for some $\eta > 1$.
(3) If $\beta \notin \mathbf{Z}$, then 1 is not (β,γ)-Liouville for any $\gamma > 1$.

The next result and proof were kindly communicated to us by Glyn Harman.

Lemma 6.8. *For any $\beta \in [0, 1)$, the set $\mathcal{L}_\beta$ of numbers which are (β,γ)-Liouville for some $\gamma > 1$ is uncountable and has zero Hausdorff dimension.*

Proof. Let $\gamma > 1$ and define $A_\gamma(n) = \gamma^{-n}/n$. Define

$$I_{m,n} = \left[\frac{m}{n} - \frac{\beta}{n} - A_\gamma(n), \frac{m}{n} - \frac{\beta}{n} + A_\gamma(n)\right].$$

Every (β,γ)-Liouville number lies in infinitely many $I_{m,n}$. Let $0 < B < C$. Since $\sum_{n=1}^{\infty} \sum_{Bn \leq m \leq Cn} |I_{m,n}|^\rho$ converges for every $\rho > 0$, it follows that the set of (β,γ)-Liouville numbers has Hausdorff dimension zero. It follows from the *countable stability* property of Hausdorff dimension, §2.2 Falconer (1990), that $\mathcal{L}_\beta$ has Hausdorff dimension zero. In particular, $\mathcal{L}_\beta$ has measure zero, for all $\beta \in \mathbf{R}$.

To show that the set of $\mathcal{L}_\beta$ is uncountable, it suffices to show that the set of (β,γ)-Liouville numbers is uncountable. Inside each interval $I_{m,n}$ there lie at least two more intervals $I_{u,w}, I_{v,w}$. For example, take $w = 2\left[\frac{n}{A_\gamma(n)}\right] + 1$ and applying this recursively we see that it contains a Cantor set homeomorphic to the usual 'middle third' Cantor set and so must be uncountable. □

From our perspective, it is more useful to quantify the set of $\beta \in [0, 1)$ for which a fixed number can be (β,γ)-Liouville for some γ. Given $x \in \mathbf{R}$ and $\gamma > 1$, define

$$\beta(x) = \{\beta \in [0, 1) \mid x \in \mathcal{L}_\beta\}, \quad \beta_\gamma(x) = \{\beta \in \beta(x) \mid x \text{ is } (\beta,\gamma)\text{-Liouville}\}.$$

Lemma 6.9. *For all $x \in \mathbf{R}$, $\beta(x)$ has zero Hausdorff dimension.*

Proof. Fix $x \in \mathbf{R}$ and for $n \in \mathbf{N}$, $\gamma > 1$ define

$$I(n,\gamma) = \left\{\beta \in [0,1) \mid \exists m \in \mathbf{Z}, \left|x - \frac{m}{n} - \frac{\beta}{n}\right| < \frac{\gamma^{-n}}{n}\right\}.$$

Clearly, $I(n,\gamma)$ is the union of at most two intervals of total length $2\gamma^{-n}$. If $\beta \in \beta(x)$, then for some $\gamma > 1$, we have $\beta \in \cup_{n \geq n_0} I(n,\gamma)$ for all n_0. Since $\Sigma_{n\geq n_0}|2\gamma^{-n}|^\rho \to 0$ as $n_0 \to \infty$ for any $\rho > 0$, it follows that $\beta_\gamma(x)$ has Hausdorff dimension zero. Choose a monotone decreasing sequence $\gamma_n \to 1$. Then $\beta(x) = \cup_{n\geq 1}\beta_{\gamma_n}(x)$ and so it follows from the countable stability property of Hausdorff dimension that $\beta(x)$ has Hausdorff dimension zero. □

Remark 6.10. It follows from lemma 6.9, that $\beta(x)$ has Lebesgue measure zero.

(*d*) *Existence of non-trivial limits*

Theorem 6.11. *Let $\lambda, \mu > 1$, $\alpha_1, \alpha_2 > 0$. There is a limit point for $(\alpha_1\lambda^n - \alpha_2\mu^m)$ if and only if $\frac{\log\lambda}{\log\mu}$ is $\left(\frac{\log\alpha_2}{\log\alpha_1 \log\mu}, \lambda\right)$-Liouville.*

Proof. Suppose that $(\alpha_1\lambda^n - \alpha_2\mu^m)$ has a limit point $L \in \mathbf{R}$. Then there exists a sequence (m_k, n_k) such that $\lim_{k\to\infty} \alpha_1\lambda^{n_k} - \alpha_2\mu^{m_k} - L = 0$. Set $\alpha = \alpha_2/\alpha_1$ and replace L by L/α_1. Dividing by λ^{n_k}, we may rewrite the limit condition as

$$1 - \alpha\frac{\mu^{m_k}}{\lambda^{n_k}} - L\lambda^{-n_k} = o(\lambda^{-n_k}).$$

We have

$$\begin{aligned}
\alpha\frac{\mu^{m_k}}{\lambda^{n_k}} &= \exp(m_k\log\mu - n_k\log\lambda + \log\alpha),\\
&= 1 + m_k\log\mu - n_k\log\lambda + \log\alpha + o(m_k\log\mu - n_k\log\lambda + \log\alpha),
\end{aligned}$$

in the limit as $k \to \infty$. Hence we may write

$$1-\alpha\frac{\mu^{m_k}}{\lambda^{n_k}}-L\lambda^{-n_k} = n_k\log\lambda - m_k\log\mu - \log\alpha - L\lambda^{-n_k} + o(m_k\log\mu - n_k\log\lambda + \log\alpha).$$

Since this expression is $o(\lambda^{-n_k})$, we have

$$\begin{aligned}
n_k\log\lambda - m_k\log\mu - \log\alpha &= L\lambda^{-n_k} + o(\lambda^{-n_k}), \text{ if } L \neq 0,\\
&= o(\lambda^{-n_k}), \text{ if } L = 0.
\end{aligned}$$

In the case $L \neq 0$, we obtain the estimate

$$\left|\frac{\log\lambda}{\log\mu} - \frac{1}{n_k}\frac{\log\alpha}{\log\mu} - \frac{m_k}{n_k}\right| \leq \frac{2L}{n_k \log\alpha \log\lambda}\lambda^{-n_k},$$

and so, since $\log\alpha = \log\alpha_2/\log\alpha_1$, $\frac{\log\lambda}{\log\mu}$ is $(\frac{\log\alpha_2}{\log\alpha_1 \log\mu}, \lambda)$-Liouville. The second case, when $L = 0$ is easier as $\frac{\log\lambda}{\log\mu}$ satisfies a stronger estimate.

For the converse, observe that if $\frac{\log\lambda}{\log\mu}$ is $(\frac{\log\alpha_2}{\log\alpha_1 \log\mu}, \lambda)$-Liouville, then the sequence $(\alpha_1\lambda^{n_k} - \alpha_2\mu^{m_k})$ is bounded and hence there is a convergent subsequence. □

Remark 6.12. Observe that if $\lambda_1 = \lambda_2$ then it follows by Example 6.7(3), that we get a non-trivial connection only if $\frac{\log\alpha_2}{\log\alpha_1 \log\mu}$ is integer valued.

Theorem 6.13. *For almost all* $(z_1, z_2) \in H_1 \times H_2$, $\omega(z_1, z_2) = \Sigma$. *In particular,* Σ *is a minimal attractor for the product system* Φ_t.

Proof. By lemma 6.3 we have $\omega(z_1, z_2) = \Sigma$ if and only if $\mathcal{A}(\{\Theta^2_{m,n}\}) = \emptyset$. It follows from (6.4) that $\mathcal{A}(\{\Theta^2_{m,n}\}) = \emptyset$ if and only if $\mathcal{A}(\{\epsilon_1\lambda_1^n - \epsilon_2\lambda_2^m\}) = \emptyset$. By theorem 6.11, $\mathcal{A}(\{\epsilon_1\lambda_1^n - \epsilon_2\lambda_2^m\}) \neq \emptyset$ if and only if $x = \log\lambda/\log\mu$ is (β, λ)-Liouville where $\beta = \log\epsilon_2/(\log\epsilon_1 \log\mu)$. Using the explicit expressions (6.5) for ϵ_1, ϵ_2, it follows easily that the fractional part of β lies in the zero measure set $\beta(x)$ for at most a zero measure set of initial conditions (z_1, z_2). Hence for almost all (z_1, z_2), $\omega(z_1, z_2) = \Sigma$ and so Σ is minimal. □

7. Discussion

In summary we have shown, under a number of technical assumptions, that products of a homoclinic attractor with a periodic or chaotic system typically lead to product attractors, but that this is not the case for products of two homoclinic attractors. We highlight some of the problems in extending our results.

Although, in theorem 6.1, we showed there were just two possibilities for the likely limit set of a product of homoclinic attractors, we were only able to identify the Milnor attractor when return times could be estimated precisely. For more general attracting homoclinic (or heteroclinic) cycles, we are so far unable to prove similar results on the structure of attractors. However, we believe that our results generalize, subject probably to generic conditions. Similar comments apply to extending our results on the product of a limit cycle or chaotic set with a homoclinic attractor.

Some of the results can clearly be generalized to the case of skew products. For example, in § 4 one could adapt the proof to allow a skew product from the periodic orbit into the attracting cycle. However, this does not seem to be straightforward for a product with a chaotic set or with another heteroclinic attractor.

The results on attractors for products of two robust cycles (§6) should be contrasted to more general problems with attractors with two-dimensional sets of connections such as described in Ashwin & Chossat (1997). In that work, the one dimensional network consisting of principal connections (namely those corresponding to the most unstable eigenvalues) were generically selected as a Milnor attractor; in our case the special product structure means that any Milnor attractor must factor to give the individual cycles.

If one of the systems has a heteroclinic attractor, asking whether all connections will be approached in the product systems is equivalent to asking whether there is nontrivial *selection of connections*; see Ashwin & Chossat (1997) and Ashwin *et al.* (2003) for examples and discussion of this effect.

Questions of cycle selection are relevant to non-transitive attractors that include connections between more general transitive (for example chaotic) subsets. For such 'cycling chaos' attractors, where the connections consist purely of one-dimensional sets, any trajectories approaching the attractor must limit to a unique connecting trajectory. In the more general case there is an issue of cycle selection. The cases where all connections are selected occur in the 'phase-resetting' and 'free-running' cases of cycling chaos investigated in Ashwin *et al.* (2003).

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