On the stability of the swirling jet shear layer

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The linear stability analysis of a simple model of a swirling jet illuminates the competition and interaction of centrifugal and Kelvin–Helmholtz instabilities. By employing potential theory, analytical expressions are derived for the growth rate and propagation velocity of both axisymmetric and helical waves. The results show that centrifugally stable flows become destabilized by sufficiently short Kelvin–Helmholtz waves. The asymptotic limits demonstrate that for long axisymmetric waves the centrifugal instability dominates, while long helical waves approach the situation of a Kelvin–Helmholtz instability in the azimuthal direction, modulated by a stable or unstable centrifugal stratification. Both short axisymmetric and short helical waves converge to the limit of a plane Kelvin–Helmholtz instability feeding on the azimuthal vorticity.

In this short note, we investigate the stability characteristics of a simple model of a swirling jet, which consists of a line vortex of strength Γ_1 at $r=0$, surrounded by a cylindrical vortex sheet at $r=R$. The unperturbed axisymmetric vortex sheet contains both azimuthal circulation (corresponding to a jump ΔU_x in the axial velocity) and streamwise circulation $\Gamma_0-\Gamma_1$ (representing a jump ΔU_θ in the circumferential velocity). The vortex lines in the sheet hence are of helical shape, with their pitch angle ψ being

$$\psi=\tan^{-1}[(\Delta U_\theta/\Delta U_x)^{1/2}]. \tag{1}$$

This model represents an extension of earlier ones investigated by Rotunno[1] and Caflisch *et al.*[2] Rotunno's work concerns the stability of the axisymmetric vortex sheet alone, i.e., without the line vortex, under axisymmetric and helical perturbations. Caflisch *et al.* investigate the linear stability with respect to axisymmetric perturbations of the purely swirling flow generated by a line vortex surrounded by a cylindrical vortex sheet containing axial circulation only. Within the current investigation, our goal is to determine if and how Rayleigh's stability criterion[3] will be modified by the addition of an axial velocity jump ΔU_x across the vortex sheet. This type of flow allows for the development of both centrifugal and Kelvin–Helmholtz instabilities, with potentially interesting interactions between the two. We employ potential theory to analyze the flow. The perturbed velocity potential is given by

$$\phi=\begin{cases}\phi_1(x,r,\theta,t)+\Gamma_1\theta/2\pi+x, & \text{inside the sheet, } r<1,\\ \phi_0(x,r,\theta,t)+\Gamma_0\theta/2\pi, & \text{outside the sheet, } r>1,\end{cases} \tag{2}$$

with the vortex sheet position $r=1+\xi(x,\theta,t)$. Here, all lengths are made dimensionless by the radius of the sheet, R, and all velocities by the streamwise velocity jump, ΔU_x. The disturbance velocity potentials satisfy $\nabla^2\phi_1=0$ and $\nabla^2\phi_0=0$ inside and outside the sheet, respectively, with the usual conditions of equal normal displacements and pressure at the sheet location. In linearized form, these read

$$\phi_{1r}=\xi_t+\Gamma_1\xi_\theta/2\pi+\xi_x \quad\text{and}\quad \phi_{0r}=\xi_t+(\Gamma_0\xi_\theta/2\pi), \tag{3}$$

$$\phi_{0t}-\xi\left(\frac{\Gamma_0}{2\pi}\right)^2+\frac{\phi_{0\theta}\Gamma_0}{2\pi}=\phi_{1t}+\phi_{1x}-\xi\left(\frac{\Gamma_1}{2\pi}\right)^2+\frac{\phi_{1\theta}\Gamma_1}{2\pi}. \tag{4}$$

By requiring the solution to be bounded on the axis and at infinity, we obtain the relevant normal mode solutions to the above equations with wave number γ,

$$\xi=e^{i(\gamma x+m\theta)+\sigma t}, \tag{5}$$

$$\phi_1=b_1I_m(\gamma r)e^{i(\gamma x+m\theta)+\sigma t}, \tag{6}$$

$$\phi_0=b_0K_m(\gamma r)e^{i(\gamma x+m\theta)+\sigma t}, \tag{7}$$

in which I_m and K_m are the modified Bessel functions, with subscripts indicating their order. In order to determine b_1, b_0, and σ, we substitute Eqs. (5)–(7) into (3) and (4). With

$$\alpha=\frac{I_m(\gamma)}{\gamma I_{m-1}(\gamma)-mI_m(\gamma)} \quad\text{and}\quad \beta=\frac{-K_m(\gamma)}{\gamma K_{m-1}(\gamma)+mK_m(\gamma)} \tag{8}$$

we thus obtain for the growth rate σ

$$\sigma=\frac{i}{\beta-\alpha}\left(\frac{m}{2\pi}(\alpha\Gamma_1-\beta\Gamma_0)+\gamma\alpha\right)\pm\Bigg[\frac{1}{4(\alpha-\beta)}\times\left(\frac{4m\gamma\alpha\Gamma_1}{\pi}+\frac{m^2}{\pi^2}(\alpha\Gamma_1^2-\beta\Gamma_0^2)+4\gamma^2\alpha-\frac{(\Gamma_0^2-\Gamma_1^2)}{\pi^2}\right)-\frac{1}{(\alpha-\beta)^2}\left(\frac{m}{2\pi}(\alpha\Gamma_1-\beta\Gamma_0)+\gamma\alpha\right)^2\Bigg]^{1/2}. \tag{9}$$

The real part of $\sigma(=\sigma_r+i\sigma_i)$ represents the nondimensional growth rate, while the imaginary part, σ_i, determines the propagation velocity, $-\sigma_i/\gamma$.

The axisymmetric case $m=0$. Using the relation $I_0(x)K_1(x)+I_1(x)K_0(x)=1/x$, we obtain $\alpha-\beta=1/[\gamma^2I_1(\gamma)K_1(\gamma)]$ which results in

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$$\sigma = -i\gamma^2 I_0(\gamma)K_1(\gamma) \pm \gamma[I_1(\gamma)K_1(\gamma)]^{1/2} \times \sqrt{I_0(\gamma)K_0(\gamma)\gamma^2 + (\Gamma_1/2\pi)^2 - (\Gamma_0/2\pi)^2}. \quad (10)$$

While the first term under the square root reflects the effect of a Kelvin–Helmholtz instability feeding on the azimuthal vorticity, the second term is related to centrifugal instability. For all $\gamma > 0$, $I_0(\gamma)$, $I_1(\gamma)$, $K_0(\gamma)$, and $K_1(\gamma)$ are $\geqslant 0$. Thus for instability to occur,

$$(2\pi\gamma)^2 I_0(\gamma)K_0(\gamma) + \Gamma_1^2 > \Gamma_0^2. \quad (11)$$

The above instability criterion (11) differs from the Rayleigh criterion by the additional term $(2\pi\gamma)^2 I_0(\gamma)K_0(\gamma)$. Hence we find that if $\Gamma_1^2 > \Gamma_0^2$, the flow remains unstable for all wave numbers (in agreement with Rayleigh's stability criterion). Note, however, that in the case of $\Gamma_1^2 < \Gamma_0^2$, the flow remains unstable to large enough wave numbers that satisfy $(2\pi\gamma)^2 I_0(\gamma)K_0(\gamma) > \Gamma_0^2 - \Gamma_1^2$. Hence a flow that is centrifugally stable can be destabilized by sufficiently short axisymmetric Kelvin–Helmholtz waves that feed on the azimuthal vorticity component of the vortex sheet. Consequently Rayleigh's criterion, which states that the flow is stable if and only if $\Gamma_1^2 < \Gamma_0^2$, no longer is a sufficient condition to guarantee stability in the swirling jet configuration.

For $\Gamma_1 = \Gamma_0 = 0$ in (10), our flow reduces to the nonswirling jet analyzed by Batchelor and Gill,[4] and we recover their solution (when recast in nondimensional form) for the growth rate and propagation velocity of an axisymmetric disturbance

$$\sigma_r = \pm\gamma[I_1(\gamma)K_1(\gamma)]^{1/2} \sqrt{I_0(\gamma)K_0(\gamma)\gamma^2} \quad (12)$$

and

$$-\sigma_i/\gamma = \gamma I_0(\gamma)K_1(\gamma). \quad (13)$$

Caflisch *et al.* consider a configuration without azimuthal vorticity, i.e., without a streamwise velocity jump ΔU_x. Hence we have to nondimensionalize differently in order to be able to compare with their results. By using Γ_1/R as the velocity scale and setting the streamwise velocity jump equal to zero, we recover their growth rate expression for vortex sheets containing axial circulation only $\sigma = \pm\gamma[I_1(\gamma)K_1(\gamma)]^{1/2}[(1/2\pi)^2 - (\Gamma_0/2\pi)^2]^{1/2}$. We now focus on the limits of long and short instability waves, respectively, i.e., on the limits of $\gamma \to 0$ and $\gamma \to \infty$. Employing the relevant asymptotic expansions for small arguments of the modified Bessel functions,[5] we obtain for $\gamma \to 0$, $I_0(\gamma) \sim 1$, $I_1(\gamma) \sim \gamma/2$, $K_0(\gamma) \sim -\ln\gamma$, $K_1(\gamma) \sim \gamma^{-1}$. Thus the expression for σ [Eq. (10)] takes the following asymptotic form for long waves:

$$\sigma \sim -i\gamma \pm \gamma/\sqrt{2}\sqrt{-\gamma^2 \ln\gamma + (\Gamma_1/2\pi)^2 - (\Gamma_0/2\pi)^2}. \quad (14)$$

For small γ the first term under the square root is positive, so that long Kelvin–Helmholtz waves act to destabilize the flow. However, the second and third term dominate for $\gamma \to 0$, since $\gamma^2 \ln\gamma \to 0$. Therefore, with $\Gamma_1^2 > \Gamma_0^2$, the swirling jet shear layer in the long-wave limit is dominated by the Rayleigh instability. The growth rate is proportional to the wave number, while the propagation velocity, $-\sigma_i/\gamma$, asymptotes to one for $\gamma \to 0$. The observation that long-wave disturbances propagate with the streamwise core velocity of the swirling jet, corresponds to Batchelor and Gill's finding for nonswirling jets. The leading-order asymptotic growth rate $\sigma_r \sim \pm(2\pi)^{-1}\gamma(\Gamma_1^2/2 - \Gamma_0^2/2)^{1/2}$ corresponds to the limiting form of the above expression derived by Caflisch *et al.* for the pure swirling flow.

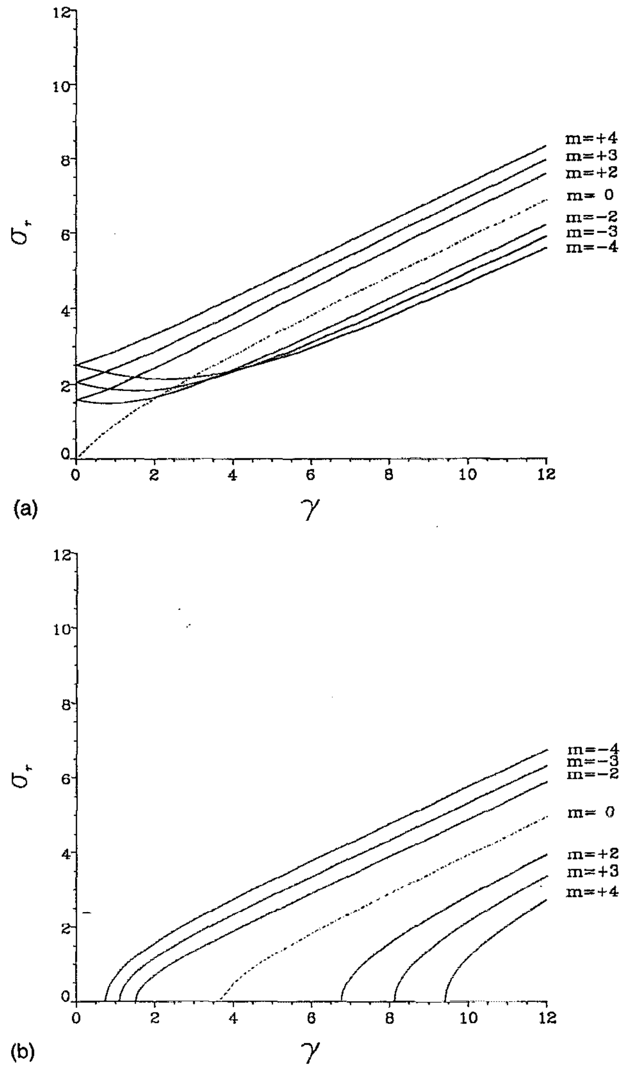

FIG. 1. Growth rate, σ_r, for $|m| = 2$, 3 and 4 with (a) $\Gamma_1 = 10.0$; $\Gamma_0 = 5.0$ (centrifugally unstable). Disturbances with positive azimuthal wave number have a higher growth rate than their negative counterparts, due to their greater alignment with the helical vortex lines. (b) $\Gamma_1 = 5.0$; $\Gamma_0 = 10.0$ (centrifugally stable). In this case the flow remains stable to long-wavelength disturbances. In both figures, growth rates for axisymmetric disturbances are indicated by dashed lines.

For very short waves, i.e., $\gamma \to \infty$, the pertinent asymptotic expansions of the modified Bessel functions lead to

$$\sigma \sim -\frac{i\gamma}{2} \pm \sqrt{\frac{\gamma^2}{4} + \frac{\gamma(\Gamma_1^2 - \Gamma_0^2)}{8\pi^2}}. \quad (15)$$

We find the overall growth rate approaching that of the plane Kelvin–Helmholtz instability alone $\sigma_r \sim \gamma/2$. The

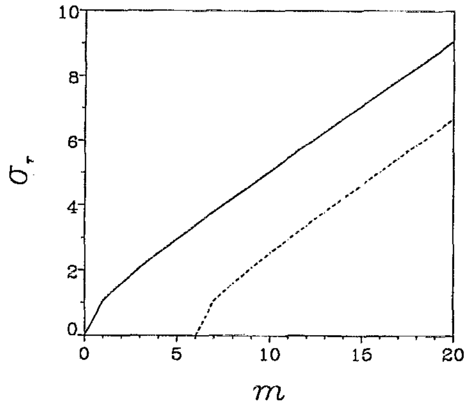

FIG. 2. Growth rate, σ_r, as a function of the azimuthal wave number for $\gamma\to 0$. Solid line: $\Gamma_1=10.0$; $\Gamma_0=5.0$; i.e., centrifugally unstable. Chain-dotted line: $\Gamma_1=5.0$: $\Gamma_0=10.0$; i.e., centrifugally stable. Notice that for this case long waves are stable.

leading-order influence of the swirl can be stabilizing or destabilizing, depending on the streamwise circulation of the line vortex and the vortex sheet.

Helical perturbations: $|m|\geqslant 2$. From the way in which we have posed the stability problem, it is obvious that only such perturbation waves can be considered that do not lead to a displacement or deformation of the centerline vortex. From symmetry reasons, it immediately follows that a helical wave $|m|=1$ violates this condition and is hence not admissible. However, all modes $|m|\geqslant 2$ maintain a symmetry of the flow field that leads to a vanishing radial velocity at the centerline.

Using Eq. (9), we show in Fig. 1(a) the growth rate versus the streamwise wave number for $|m|=2$, 3, and 4 when $\Gamma_1=10.0$ and $\Gamma_0=5.0$, i.e., when the flow is centrifugally unstable. The graph reflects the broken symmetry between positive and negative values of m as a result of the streamwise vorticity in the vortex sheet. All helical modes have positive growth rates for all values of γ. In Fig. 1(b) we show the corresponding centrifugally stable case with $\Gamma_1=5.0$ and $\Gamma_0=10.0$.

Again the most interesting limits concern $\gamma\to 0$ and $\gamma\to\infty$, while keeping m fixed. For $\gamma\to 0$, the small argument expansions for the modified Bessel functions yield $\alpha\to 1/m$, $\beta\to -1/m$, so that we obtain for the complex growth rate

$$\sigma\sim -\frac{im}{4\pi}(\Gamma_1+\Gamma_0) \pm(2\pi)^{-1}\sqrt{\frac{m^2}{4}(\Gamma_1-\Gamma_0)^2+\frac{m}{2}(\Gamma_1^2-\Gamma_0^2)} \qquad (16)$$

for $m\geqslant 0$. This limit $\gamma\to 0$ corresponds to a situation in which the wave-number vector becomes aligned with the azimuthal direction. In this case, the flow can develop a Kelvin–Helmholtz instability, feeding on the streamwise vorticity component. For $\gamma\to 0$, Fig. 2 shows the real part of the growth rate as a function of m both for a centrifugally unstable and a centrifugally stable stratification. In the limit of large m, $\sigma_r\propto m$, as the curvature of the shear layer becomes less important. It should be pointed out that the presence of the centerline vortex is important for the real part of the growth rate, as it affects the centrifugal stratification. The imaginary part of the growth rate depends on the presence of the centerline vortex as well. However, this effect is a purely kinematic one, as the rotation of the helical wave by the centerline vortex leads to a streamwise propagation velocity of the wave crest in the plane θ=const. It is clear from Eq. (16) that this streamwise propagation velocity approaches infinity as $\gamma\to 0$.

In the limit of fixed m and $\gamma\to\infty$, we obtain a situation that is quite different. Using the large argument expansions for the modified Bessel functions, we obtain $\alpha\to 1/\gamma$, $\beta\to -1/\gamma$, and σ asymptotically approaches $\sigma\sim -i\gamma/2\pm\gamma/2$, i.e., the growth rate of the plane Kelvin–Helmholtz instability.

The present model of a swirling jet lends itself well to a nonlinear and three-dimensional investigation using vortex methods. A study along these lines is currently underway in order to compare the three-dimensional evolution of swirling jets with that of nonswirling ones (Martin and Meiburg[6,7]).

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