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An Analytical Model for the Effect of Elastic Modulus Mismatch on Laminate Threshold Strength

Alok Paranjpye, Glenn E. Beltz and Noel C. MacDonald
Department of Mechanical and Environmental Engineering,
University of California, Santa Barbara
Santa Barbara, CA 93106, U.S.A

ABSTRACT

A scheme for calculating the stress intensity factor at the crack tip in a two-dimensional bimaterial laminate composite has been developed to model the damage tolerance obtained in brittle ceramics by using this geometry. One limitation of the model is its assumption of homogenous elastic properties throughout the composite, limiting the accuracy of predictions it can make about real material systems. Finite element simulations of the same architecture that allow for elastic modulus mismatch give results that are moderately different from those obtained from the homogeneous model. We present an analytical expression for the stress intensity factor around a crack tip in a laminated composite that can take into account the elastic modulus mismatch. To make the problem tractable, the model is based on the assumption that the system behaves as a homogeneous anisotropic material when the stress field at the crack tip arises out of far field tractions applied away from the crack tip. The model improves upon the homogeneous model, giving results that are closer to those from the finite element simulations. We, however, conclude that more work is required to predict the stresses at the tip as the crack approaches a material interface before a complete analytical model can be obtained.

INTRODUCTION

A bimaterial laminate architecture (Figure 1) comprised of alternating layers of residual biaxial compressive and tensile stress has been shown [1] to be effective in arresting the propagation of cracks through the structure. This scheme, when applied to ceramics or other brittle materials, effectively changes the strength distribution of these materials from probabilistic to deterministic in a specific applied stress range. The material develops a threshold strength below which it does not fail, and above which the strength is determined by the size of the largest flaw in the structure. A stress intensity function that incorporates the effects of the alternating residual stress was developed to describe the shielding of the crack tip in the compressive layers from the external applied stress,

$$K = \sigma_a\sqrt{\pi a} + \sigma_c\sqrt{\pi a}\left[\left(1+\frac{t_1}{t_2}\right)\frac{2}{\pi}\sin^{-1}\left(\frac{t_2}{2a}\right)-1\right] \tag{1}$$

where σ_a is the applied stress, a is the half crack length, t_1 and t_2 are the thickness of the compressive and tensile layers. σ_c is the magnitude of the residual biaxial stress in the compressive layers and is evaluated from [2] as

$$\sigma_c = \varepsilon E_1^{'}\left(1+\frac{t_1E_1^{'}}{t_2E_2^{'}}\right)^{-1} \tag{2}$$

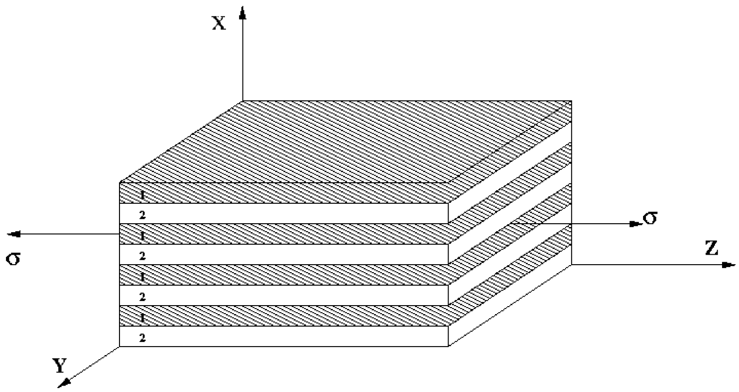

Figure 1. Bimaterial laminate geometry: 2-3 is the plane of transverse isotropy

where ε is the residual differential thermal strain, $E_i^{'} = E_i/(1-\nu_i)$ is the modified Young's modulus for each material component, and ν is Poisson's ratio. Equation (1) shows that the stress intensity factor at the crack tip is reduced as a larger fraction of the crack length lies in the two compressive layers on either side of the tensile layer. Setting K to the critical stress intensity factor of the compressive layer material, $2a = t_2 + 2t_1$ and rearranging terms, we can obtain an expression for a threshold stress below which a crack cannot extend through the compressive layers to cause failure.

$$\sigma_{thr} = \frac{K_c}{\sqrt{\pi \frac{t_2}{2}\left(1+\frac{2t_1}{t_2}\right)}} + \sigma_c\left[1-\left(1+\frac{t_1}{t_2}\right)\frac{2}{\pi}\sin^{-1}\left(\frac{t_1}{t_2+2t_2}\right)\right] \tag{3}$$

Equation (3) can be used to predict the efficacy of this scheme towards obtaining predictable strengths in brittle materials. The threshold strength increases with the magnitude of the residual compressive stress in alternate layers, and varies inversely with the size scale of the layer thickness. The threshold strength effect has also been validated in experimental studies in an alumina/mullite system [1,3] as well as in silicon/silica system with much smaller layer thickness [4]. This expression can also be utilized to optimize the threshold stress that may be obtained in a given material system by adjusting the layer thickness ratio. Work by McMeeking and Hbaieb [5] has shown that an optimal threshold strength for low toughness materials of approximately $0.3\varepsilon E_1^{'}$ may be obtained by keeping the layer thickness ratio in the range 1-2.8.

Equations (1) and (3) approximate the bimaterial laminate as an elastically homogenous system in evaluating the stress intensity factor at the crack tip, and the resulting threshold stress. As such, all predictions made from these equations are limited in their accuracy when the Young's moduli and Poisson's ratio values of the two materials in the laminate are significantly different. Hbaieb and McMeeking [6] have conducted a finite element simulation of this geometry, and presented extensive results on the variation of threshold stress with varying material modulus ratios. Figure 2 shows a comparison of the results from the finite element model with the predictions from the analytical expression. The results start to differ greatly as the modulus ratio E_1/E_2 varies from unity.

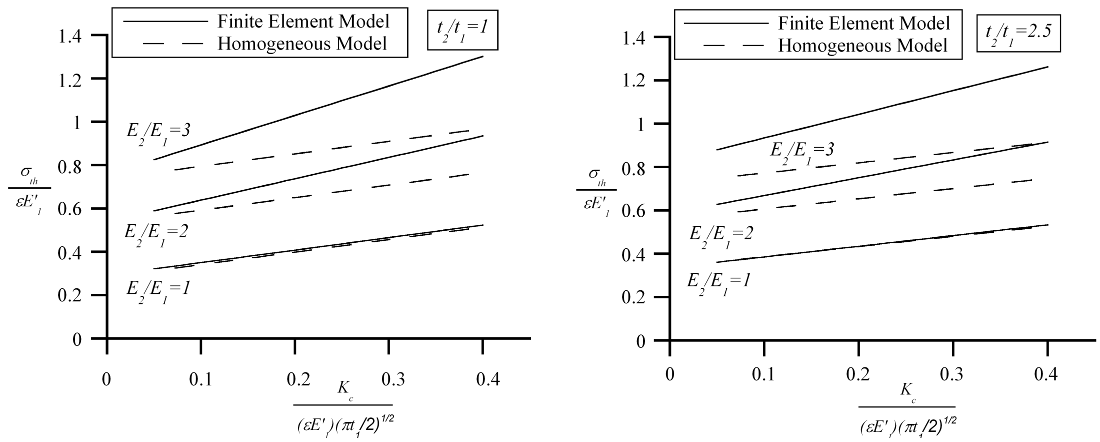

Figure 2. Comparison of results from Analytical model and FEA for two layer width ratios

The finite element results provide a great deal of insight into the crack arrest phenomenon due to the presence of a compressive stress and its effectiveness with varying material parameters, much more so than an analytical model assuming material homogeneity. The drawback of the finite element analysis is that it cannot be applied very easily towards optimization of material and geometric parameters towards obtaining the highest possible threshold stress, in the way that a closed form analytical expression can be. For this reason, we seek to obtain an alternate model for predicting the crack tip stress intensity and threshold stress that can take the material inhomogeneity into account and yield a closed form expression.

ORTHOTROPIC LAMINATE

To arrive at an analytical expression for the stress intensity factor at the crack tip, we treat the bimaterial laminate as an anisotropic material in a limited sense. We assume that the stresses building up at the crack tip due to stresses applied at the material boundary are similar in our composite as they would be for a crack tip of identical geometry in an *anisotropic* material. Tada et al. [7] give an expression relating the energy release rate at the crack tip in an anisotropic material to the stress intensity factor K_i for the corresponding isotropic boundary value problem

$$G_i = \boldsymbol{C} K_i^2 \tag{4}$$

where $\boldsymbol{C}$ is computed from the elements of the compliance matrix for the anisotropic material. For the case of an bimaterial laminate, the material is orthotropic (specifically, transversely isotropic), and for a crack opening in Mode 1, $\boldsymbol{C}$ is evaluated as

$$\boldsymbol{C} = \sqrt{\frac{A_{11}A_{22}}{2}}\left[\sqrt{\frac{A_{22}}{A_{11}}} + \frac{2A_{12}+A_{66}}{2A_{11}}\right]^{1/2} \tag{5}$$

where A_{ij}'s are the elements of the compliance matrix. The stress intensity factor for the isotropic boundary value problem is that evaluated by Rao [1], which takes into account the effect of the residual stresses on the crack tip, and which we will refer to as $K_1^{isotropic}$.

The energy release rate thus obtained can be related back to a crack tip stress intensity factor in an anisotropic material as

$$K_1^{anisotropic} = \sqrt{\frac{E}{(1-\nu^2)}} G_1^{1/2} \tag{6}$$

for the plane strain situation that we will consider. The Young's modulus and Poisson's ratio in (6) are for the material at the crack tip, which in our case is the compressive layer material. From (4) and (6), we obtain

$$\boldsymbol{F} = \frac{K_1^{anisotropic}}{K_1^{isotropic}} = \sqrt{\frac{E\boldsymbol{C}}{(1-\nu^2)}} \tag{7}$$

<u>Constitutive Relations for an Orthotropic Laminate</u>

To obtain the compliance matrix for an orthotropic laminate with transverse isotropy in the 2-3 plane, we solve the general elasticity problem as shown in Figure 1. We apply stresses along the material axes of isotropy and compute the strains. The ratio of the strains to the stresses gives the elements of the elasticity matrix, related to the material constants of the individual components of the laminate and their relative thickness. For a transversely isotropic material there are only 5 independent elastic constants, and the compliance matrix is

$$\begin{bmatrix} A_{11} & A_{12} & A_{13} & 0 & 0 & 0 \\ A_{21} & A_{22} & A_{23} & 0 & 0 & 0 \\ A_{31} & A_{32} & A_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & A_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & A_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & A_{66} \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{12} & 0 & 0 & 0 \\ S_{12} & S_{22} & S_{23} & 0 & 0 & 0 \\ S_{12} & S_{23} & S_{22} & 0 & 0 & 0 \\ 0 & 0 & 0 & 2(S_{22}-S_{23}) & 0 & 0 \\ 0 & 0 & 0 & 0 & S_{66} & 0 \\ 0 & 0 & 0 & 0 & 0 & S_{66} \end{bmatrix} \tag{8}$$

where

$$S_{11} = \frac{E_B^2 t_A t_B (1-\nu_A - 2\nu_A^2) + E_A E_B (t_B^2(1-\nu_A) + t_A^2(1-\nu_B) + 4t_A t_B \nu_A \nu_B) + E_A^2 t_A t_B (1-\nu_B - 2\nu_B^2)}{E_A E_B (t_A + t_B)[E_B t_B (1-\nu_A) + E_A t_A (1-\nu_B)]} \tag{9}$$

$$S_{12} = \frac{t_A \nu_A (1-\nu_B) + t_B \nu_B (1-\nu_A)}{E_B t_B (1-\nu_A) + E_A t_A (1-\nu_B)} \tag{10}$$

$$S_{23} = \frac{(t_A + t_B)[E_B t_B \nu_B (1-\nu_A^2) + E_A t_A \nu_A (1-\nu_B^2)]}{E_B^2 t_B^2 (1-\nu_A^2) + 2E_A E_B t_A t_B (1-\nu_A \nu_B) + E_A^2 t_A^2 (1-\nu_B^2)} \tag{11}$$

$$S_{22} = \frac{(t_A + t_B)[E_B t_B (1-\nu_A^2) + E_A t_A (1-\nu_B^2)]}{E_B^2 t_B^2 (1-\nu_A^2) + 2E_A E_B t_A t_B (1-\nu_A \nu_B) + E_A^2 t_A^2 (1-\nu_B^2)} \tag{12}$$

$$S_{44} = 2(S_{22} - S_{23}) = \frac{2(t_A + t_B)(1-\nu_A)(1-\nu_B)}{E_B t_B (1+\nu_A) + E_A t_A (1+\nu_B)} \tag{13}$$

$$S_{66} = \frac{E_B t_A (1+\nu_A) + E_A t_B (1+\nu_B)}{E_A E_B (t_A + t_B)} \tag{14}$$

The factor $\boldsymbol{F}$ relating $K_1^{anisotropic}$ to $K_1^{isotropic}$ is calculated using (9)-(14). The expression is cumbersome, does not in itself offer any new insights, and is not presented here. However, it has been verified to be equal to unity when the condition $E_A = E_B$ and $\nu_A = \nu_B$ is applied. The expression for threshold stress for this model is

$$\sigma_{thr} = \sqrt{\frac{(1-\nu_1^2)}{E_1 \boldsymbol{C}}} \frac{K_c}{\sqrt{\pi \frac{t_2}{2}\left(1+\frac{2t_1}{t_2}\right)}} + \sigma_c \left[1 - \left(1+\frac{t_1}{t_2}\right)\frac{2}{\pi}\sin^{-1}\left(\frac{t_1}{t_2+2t_2}\right)\right] \tag{15}$$

RESULTS

The expression obtained above predicts a higher value of threshold strength for the brittle ceramic composite. We compare it to the results of the finite element simulation to as well as the predictions from the homogenous model. It is worthwhile to note that our analysis affects only the first term of the original result for threshold strength (1). The first term is independent of any residual stresses in the system, instead reflecting the influence of the critical stress intensity factor of the compressive layer material, and more importantly of the layer width size scale. Figure 3 shows a comparison of the predicted threshold stress from the finite element simulation, the isotropic material model, and the model described above with varying modulus ratios.

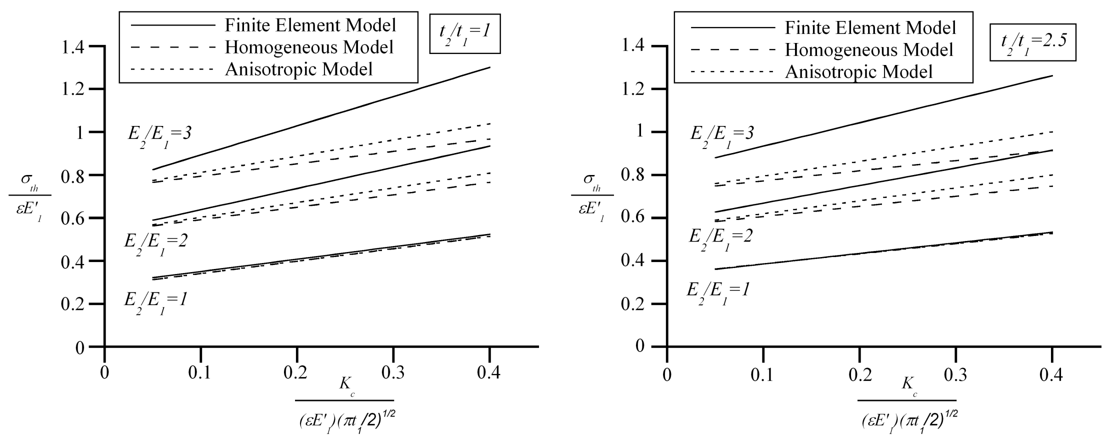

Figure 3. Comparison of results from FEM and two modeling methods

The FEM results are seen to differ increasingly from the homogenous model results as the modulus mismatch is increased, as well as for higher values for critical stress intensity of the compressive layer material. The anisotropic model bridges the gap somewhat, and reflects the effect of the modulus mismatch on the rate of change of threshold strength with critical stress intensity factor. The y-intercepts of the curves in Figure 3 are not influenced by the modifications to the homogenous model, and lead to an underestimation of the threshold stress at low fracture toughness. Figure 4 shows the influence of the elastic modulus ratio on threshold stress.

The threshold stress achievable drops sharply as the tensile layer material becomes softer than the compressive layer material, for all values of K_c for the compressive layer. For E_2/E_1 greater than 1, σ_{thr} quickly becomes constant for relatively low toughness materials, while it continues to rise significantly for higher compressive layer toughness values.

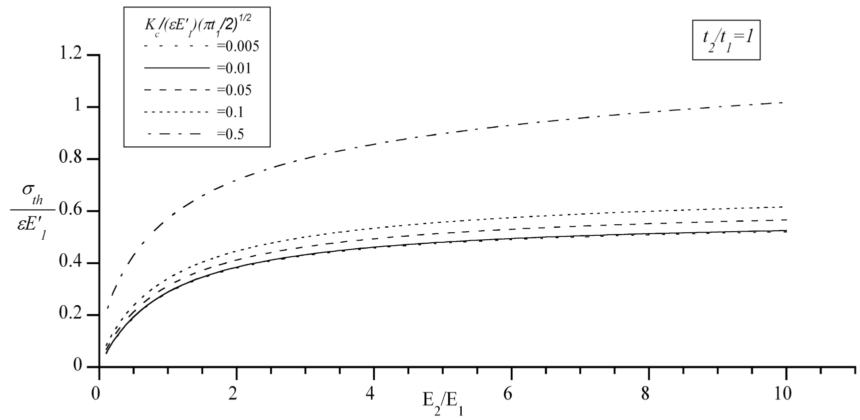

Figure 4. Threshold stress dependence on modulus ratio

CONCLUSIONS

The modeling strategy presented here represents an important step forward towards optimization of parameters for achieving the highest possible threshold stress in a bimaterial laminate geometry in a given material system. There is a significant discrepancy in results between the homogenous analytical model and the finite element analysis when the elastic constant of the two materials do not match, and this becomes more significant as the layer size scale is reduced. Ongoing work is being conducted towards applying this framework to improving the reliability of MEMS structures involves sub-micron layer widths. Optimizing the layer width ratio in this study required an improved analytical model, since using finite element simulations towards the same end is relatively difficult. The modifications presented here modestly address this need. However, this model remains only an approximation to an exact formulation for a bimaterial laminate system. The challenge lies in accurately modeling the stresses as the crack tip approaches the interface between a compressive and a tensile layer, since it is this situation that ultimately determines the threshold stress. The current model has a shortcoming in not making any allowance for the crack tip approaching an interface. We are working on developing an understanding of this situation, and developing a more complete and accurate description of cracking in bimaterial laminates.

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