Myths, Misconceptions and Misuses of Nonlinearity

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DoyleFest, July ’04

The Issues

Observed Phenomena · · · · · · · · · · Mathematical Models

- Phenomena: Data, system trajectories, time traces
  Models: Differential equations

- Concern: Complexity
  - Complexity of phenomena (scale, patterns, etc.)

    somewhat subjective
  - Complexity of models (?????)

    Linear vs. Nonlinear

    Low vs. High order

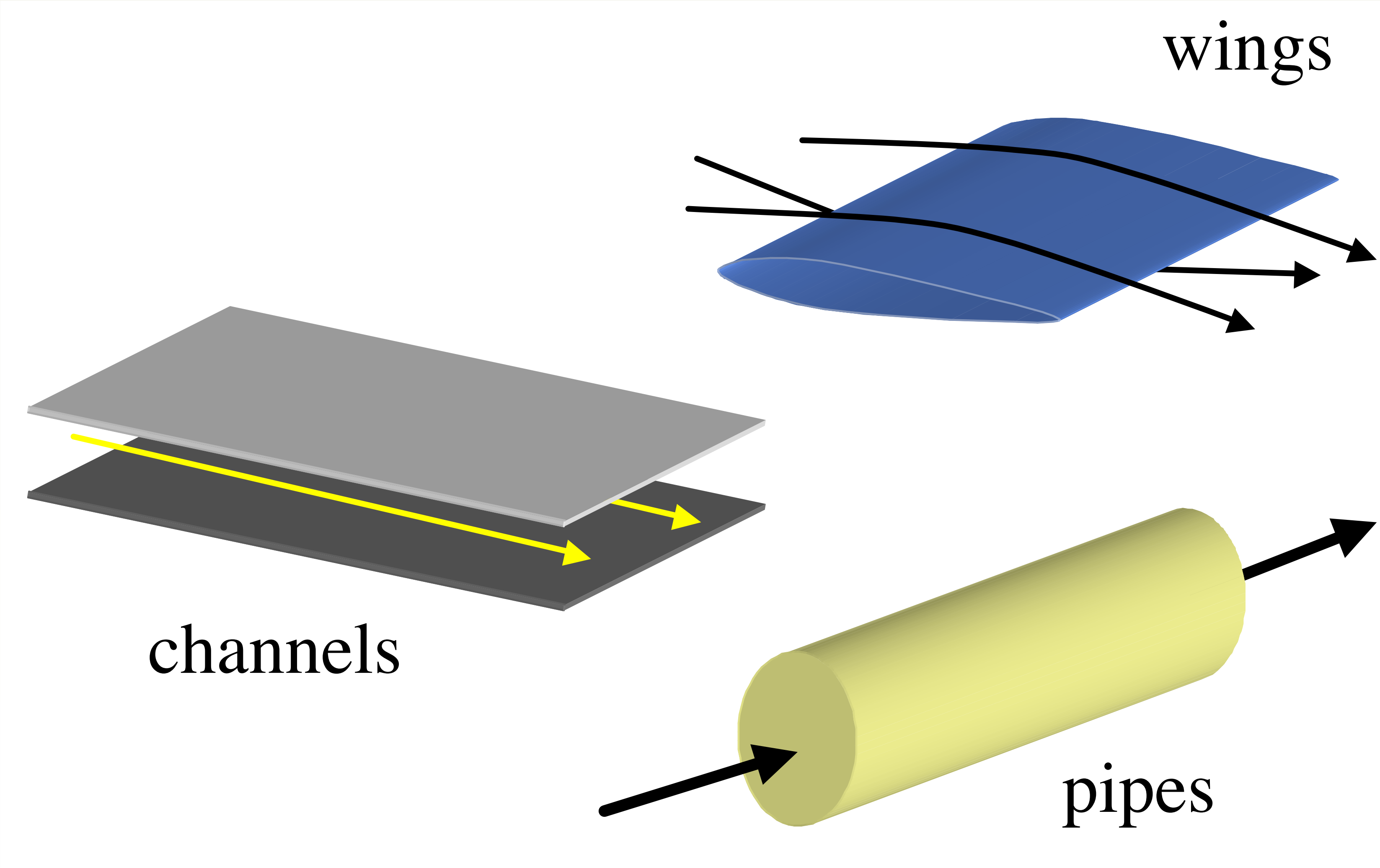
    . . . .

Outline

- A story of boundary layer turbulence
  - Effective analysis using tools from linear systems theory
  - "but I thought turbulence was a nonlinear phenomenon!!??"
- Are linear and nonlinear models equivalent?
  - Linearization techniques
  - Model order
  - Special structure
- The use of "nonlinearity" in current scientific culture
  - a runaway concept

The phenomenon of turbulence

Technologically important flows: *Flows past streamlined bodies*

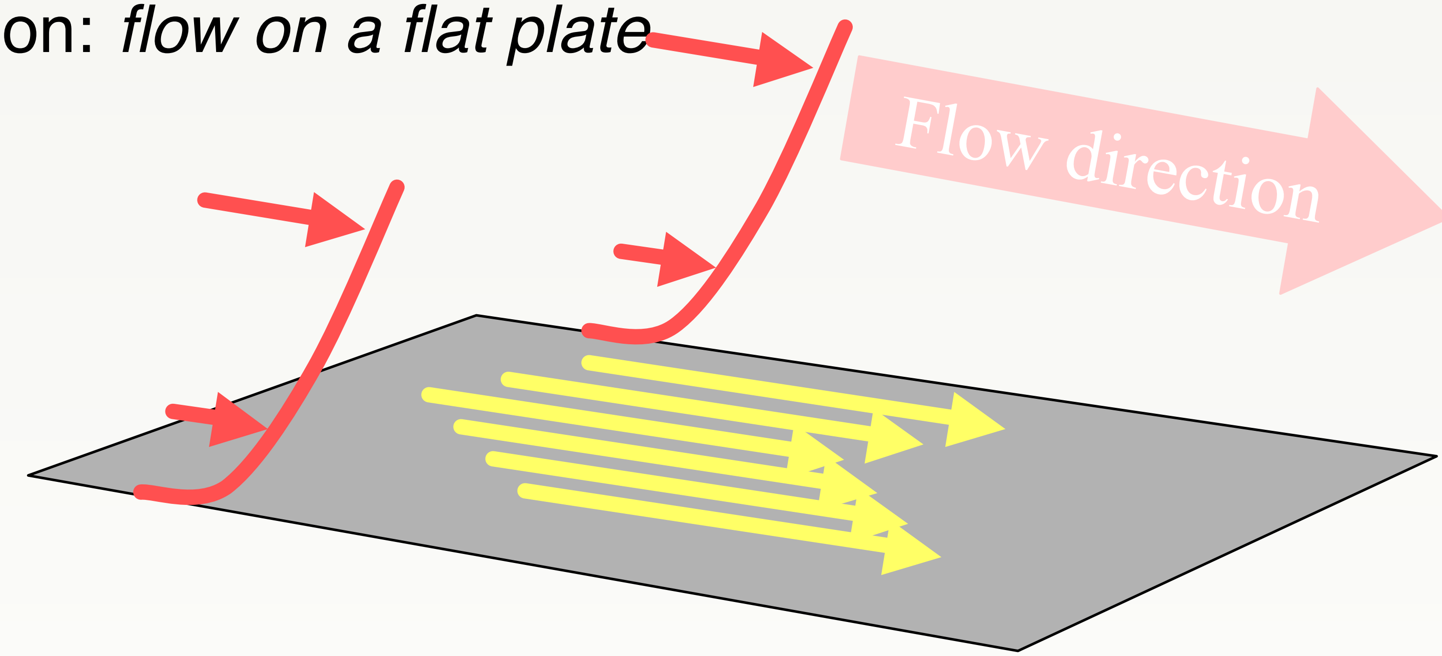


Wall-bounded shear flows *Friction with the walls drives the flows*

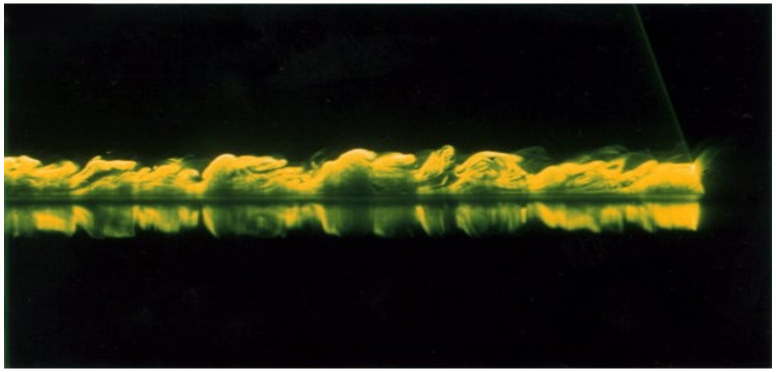
The phenomenon of turbulence (Cont.)

Boundary layers form in flow past any surface

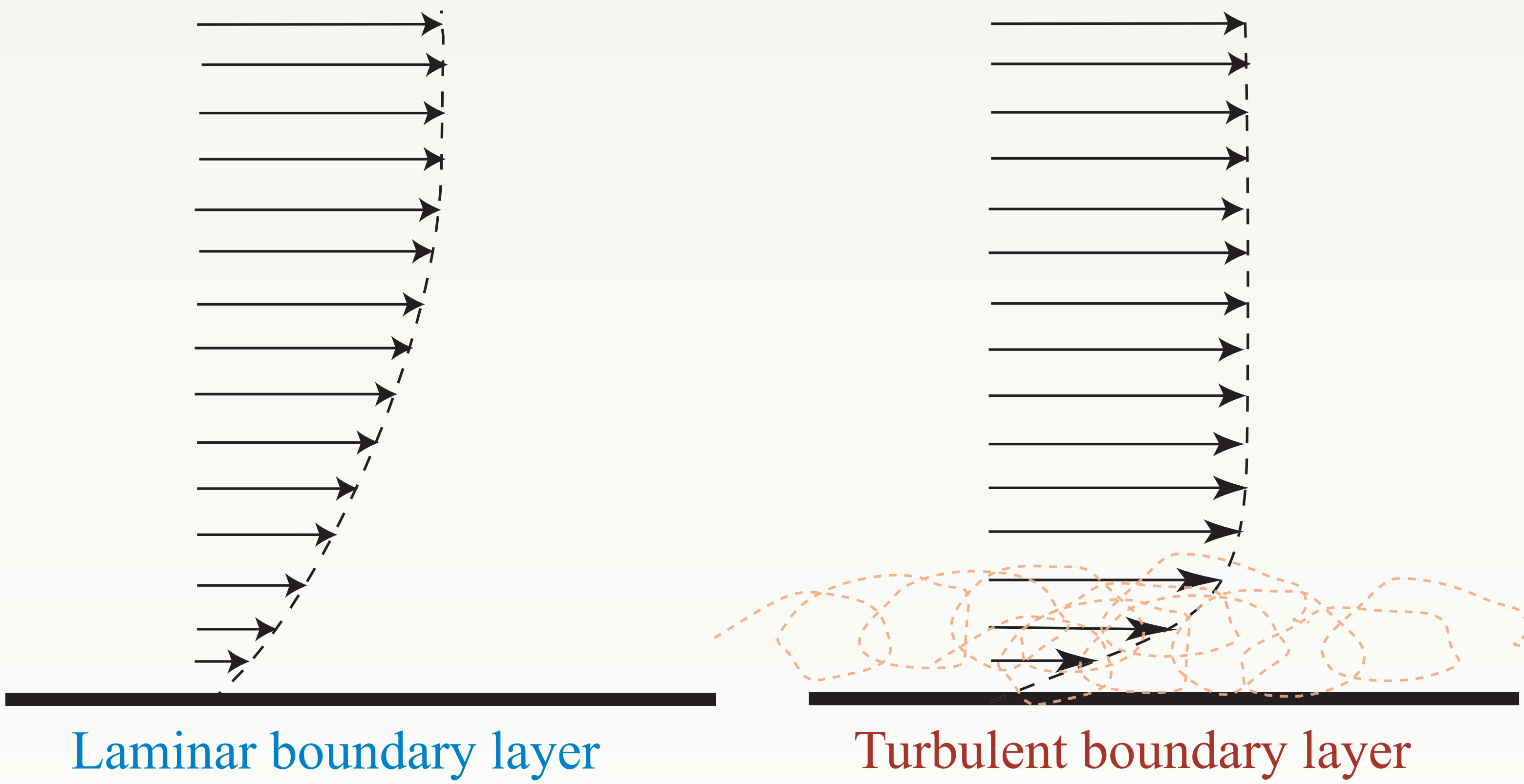
Idealization: *flow on a flat plate*



Viewed sideways



Boundary layer turbulence and skin-friction drag

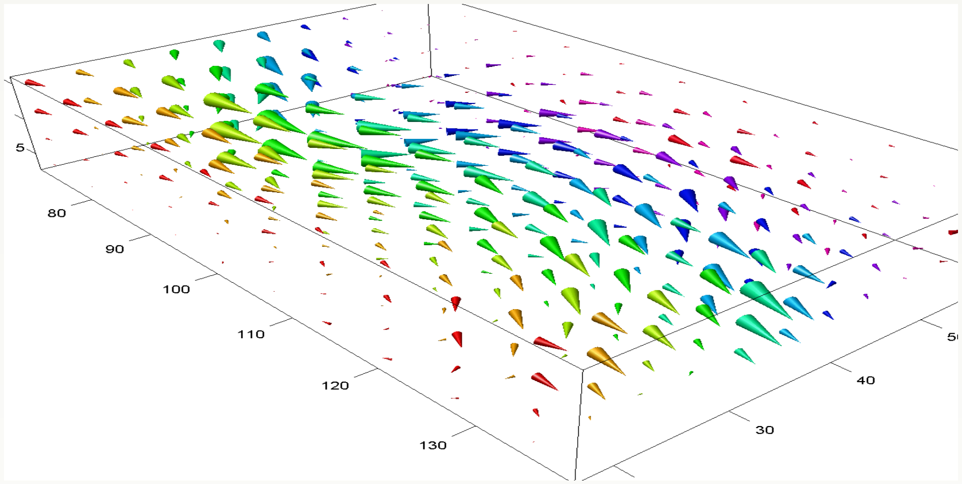


A laminar BL causes less drag than a turbulent BL (for same free-stream velocity)

This *skin-friction drag* is 40-50% of total drag on typical airliner

The "Dynamical Systems" view

The flow field at time time $=: \quad \mathbf{\Psi}(t)$



$\mathbf{\Psi}(t)$ is the vector field of flow velocities

$\mathbf{\Psi}(t)$ is the STATE of the system at time t

Fluid dynamics (e.g. *Navier-Stokes equations*) can be written as:

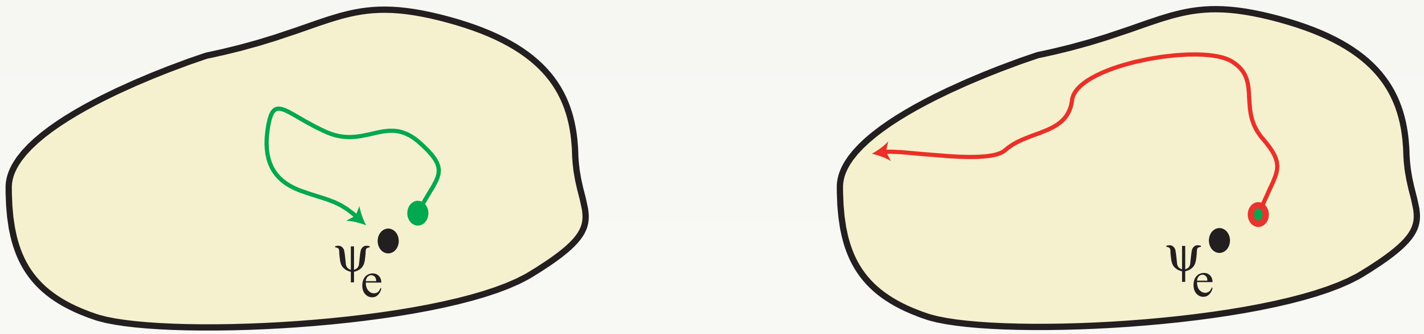
$$\partial_t \mathbf{\Psi}(t) \qquad = \quad \mathcal{F}\Big(\mathbf{\Psi}(t), R\Big)$$

$\uparrow$ change in time

$\uparrow$ function of current state

Does the system stay near an equilibrium?

If starting “near” equilibrium, does system come back to it??



stable equilibrium unstable equilibrium

An unstable equilibrium is not really an “equilibrium”

How to check stability?

Common method: Linearization

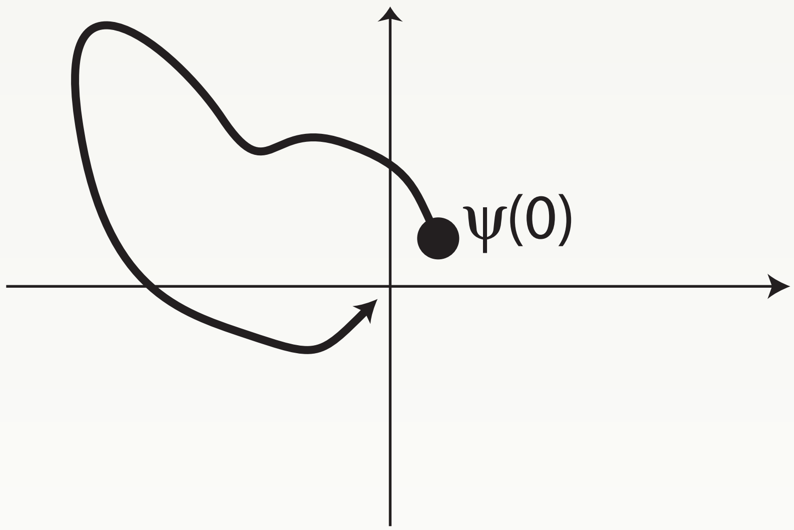
$$\partial_t \mathbf{\Psi}(t) \;=\; \mathcal{F}\Big(\mathbf{\Psi}(t)\Big) \;=\; \underset{\substack{\uparrow \\ \text{Linear part}}}{\mathcal{A}}\Big(\mathbf{\Psi}(t)\Big) + \underset{\substack{\uparrow \\ \text{The rest}}}{\mathcal{N}}\Big(\mathbf{\Psi}(t)\Big)$$

Stability (instability) of $\mathcal{A}$ $\Leftrightarrow$ Local stability (instability) of $\mathcal{F}$.

Can be studied using eigenvalue/eigenfunction analysis of $\mathcal{A}$

Uncertainty in a Dynamical System

Lyapunov Stability deals with **uncertainty** in initial conditions



If $\boldsymbol{\Psi}(0)$ is known to be *precisely* $\boldsymbol{\Psi}_e$, then $\boldsymbol{\Psi}(t) = \boldsymbol{\Psi}_e,\ t \geq 0$

We introduce the concept of Lyapunov stability because we can never be *infinitely certain* about the initial condition

Shortcomings of Lyapunov stability
- Perturbs only initial conditions
- Cares only about asymptotic behavior

Uncertainty in a Dynamical System (cont.)

Lyapunov stability

$$\dot{\psi} = f(\psi)$$

uncertain initial conditions

investigate $\lim_{t\to\infty} \psi(t)$

investigate transients

e.g. $\sup_{t\geq 0} \|\psi(t)\|$

dynamical uncertainty

$$\dot{\psi} = F(\psi) + \Delta(\psi)$$

exogenous disturbances

$$\dot{\psi}(t) = F(\psi(t), d(t))$$

combinations

$$\dot{\psi}(t) = F(\psi(t), d(t)) + \Delta(\psi(t), d(t))$$

$\Longrightarrow \Longrightarrow \Longrightarrow \Longrightarrow \Longrightarrow \Longrightarrow \Longrightarrow \Longrightarrow \Longrightarrow$ **More uncertainty** $\Longrightarrow$

Linearized version:

transient growth

eigenvalue stability

$$\dot{\psi} = A\psi$$

umodelled dynamics

$$\dot{\psi} = (A + \Delta)\psi$$

Psuedo-spectrum

exogenous disturbances

$$\dot{\psi}(t) = A\psi(t) + Bd(t)$$

input-output analysis

combinations

$$\dot{\psi}(t) = (A + B\Delta C)\psi(t) + (F + G\Delta H)d(t)$$

Uncertainty in a Dynamical System (cont.)

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input-output analysis

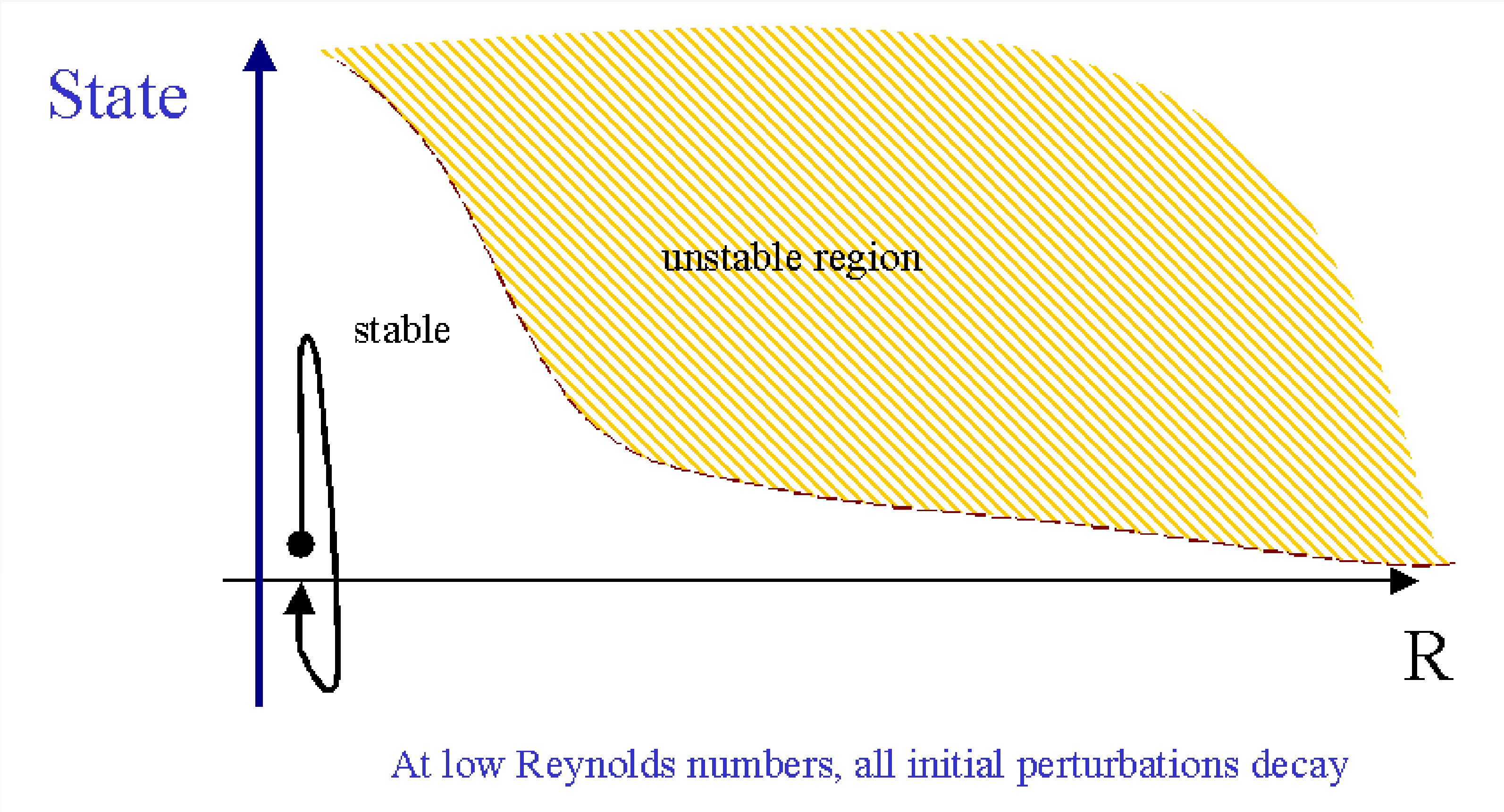
The nature of turbulence

Fluid dynamics are described by deterministic equations

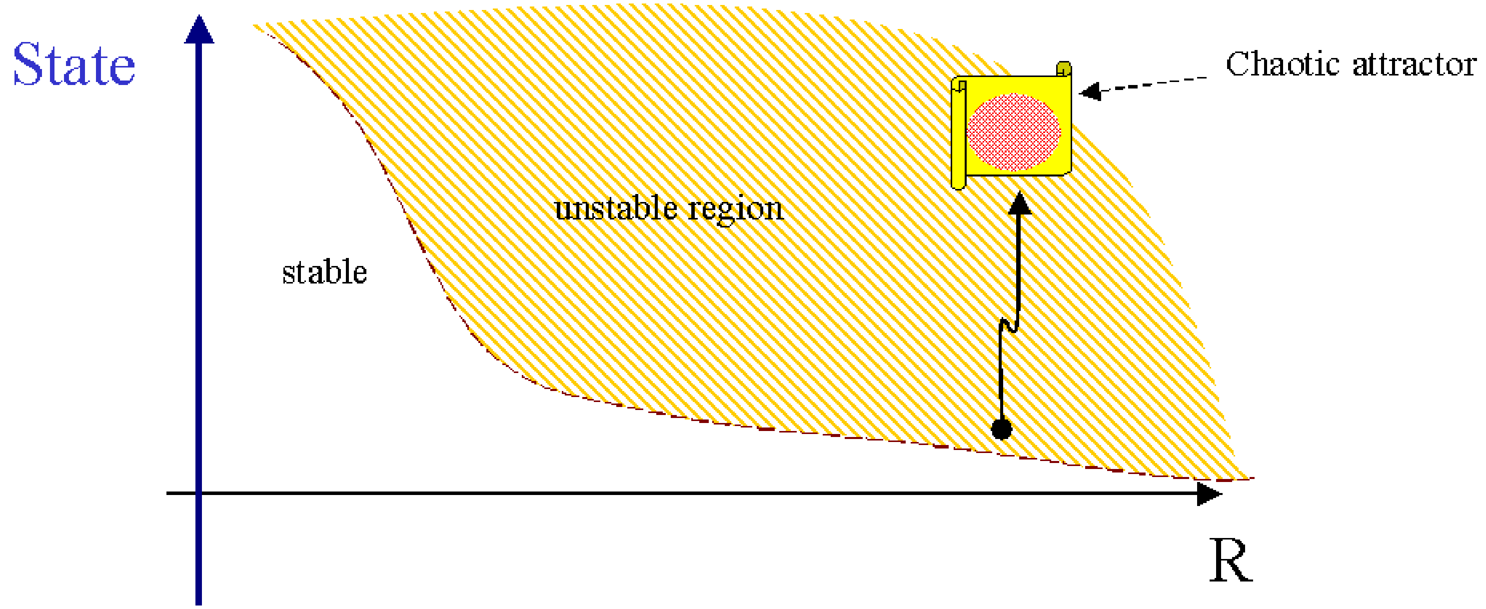
Why does fluid flow “look random” at high Reynolds numbers??

The nature of turbulence

Common view of turbulence



Common view of turbulence



At high Reynolds numbers, move away from laminar flow,
Then flow looks turbulent because it evolves on a chaotic attractor

Intuitive reasoning

Complex, statistical looking behavior ⟷ System with chaotic dynamics

The nature of turbulence (cont.)

Alternative view

Noise
Surface roughness
Thermal forces
Free stream disturbances

→ **Dynamics of the Navier-Stokes Equations** →

Fluctuating flow field

(looks statistical)

Qualitatively similar to

Noise
Surface roughness
Thermal forces
Free stream disturbances

→ **Dynamics of the *Linearized* Navier-Stokes Equations** →

Fluctuating flow field

(looks statistical)

So what now?

- A story of boundary layer turbulence
  - Effective analysis using tools from linear systems theory
  - "but I thought turbulence was a nonlinear phenomenon!!??"
- Are linear and nonlinear models equivalent?
  - Linearization techniques
  - Model order
  - Special structure
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Linearization techniques

The Carleman Linearization

Example: $\frac{d}{dt}x \;=\; x^2$

Define: $x_1 := x, \;\; x_2 := x^2, \;\; , \cdots , \; x_n := x^n, \;\; \cdots$

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ \vdots \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & \\ 0 & 0 & 2 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 3 & 0 & \\ 0 & 0 & 0 & 0 & 4 & \\ & & \vdots & & & \ddots \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ \vdots \end{bmatrix}, \qquad \text{i.e.} \qquad \dot{\mathcal{X}} \;=\; \mathcal{A}\mathcal{X}$$

The original system is imbedded in this linear system

General procedure: Given $\dot{x} \;=\; F(x, u)$

Define $\mathcal{X}$ with components $x^i u^j$

$$\dot{\mathcal{X}} \;=\; \mathcal{A}\mathcal{X} \;+\; (\dot{u})\, \mathcal{B}\mathcal{X}$$

Thus: *Bilinear Systems are Universal Models*

Linearization techniques (cont.)

- **The Lie-Koopman Linearization**
- **State transformations** and **feedback linearization**
- **The Fokker-Planck equation**
- $\cdots\cdots\cdots$

Linearization techniques often do not make a problem more tractable

e.g. computing $e^{t\mathcal{A}}$ can be arbitrarily complex!!

Linear Phenomena

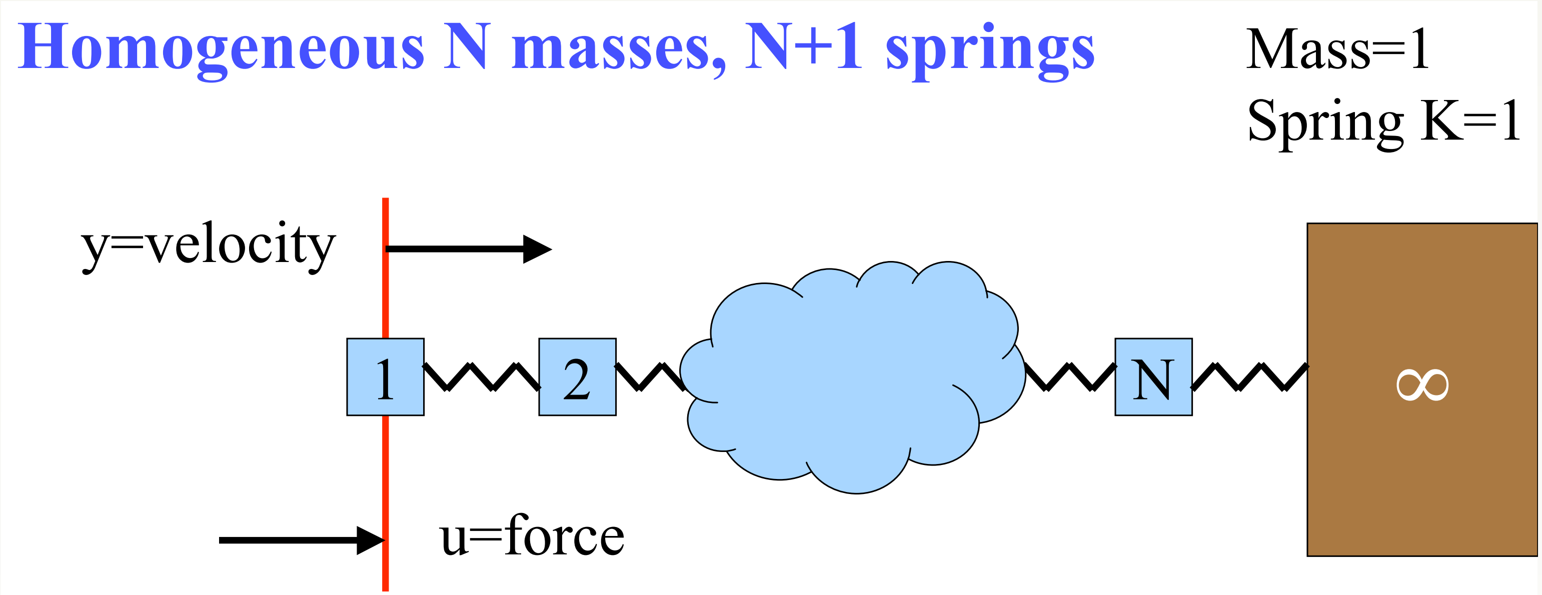
Q: What phenomena can a linear dynamical system explain?

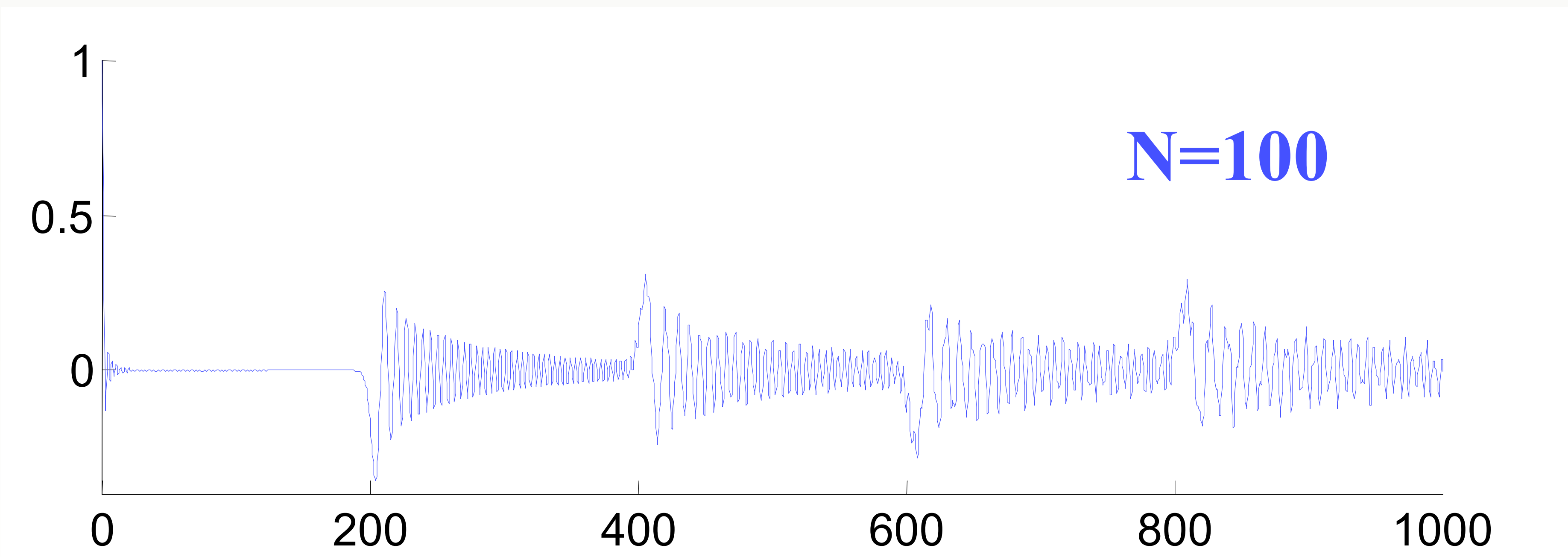
$$\frac{d}{dt}\Psi(t) = \mathcal{A}\,\Psi(t)$$

A: Any phenomena explainable using a nonlinear dynamical system

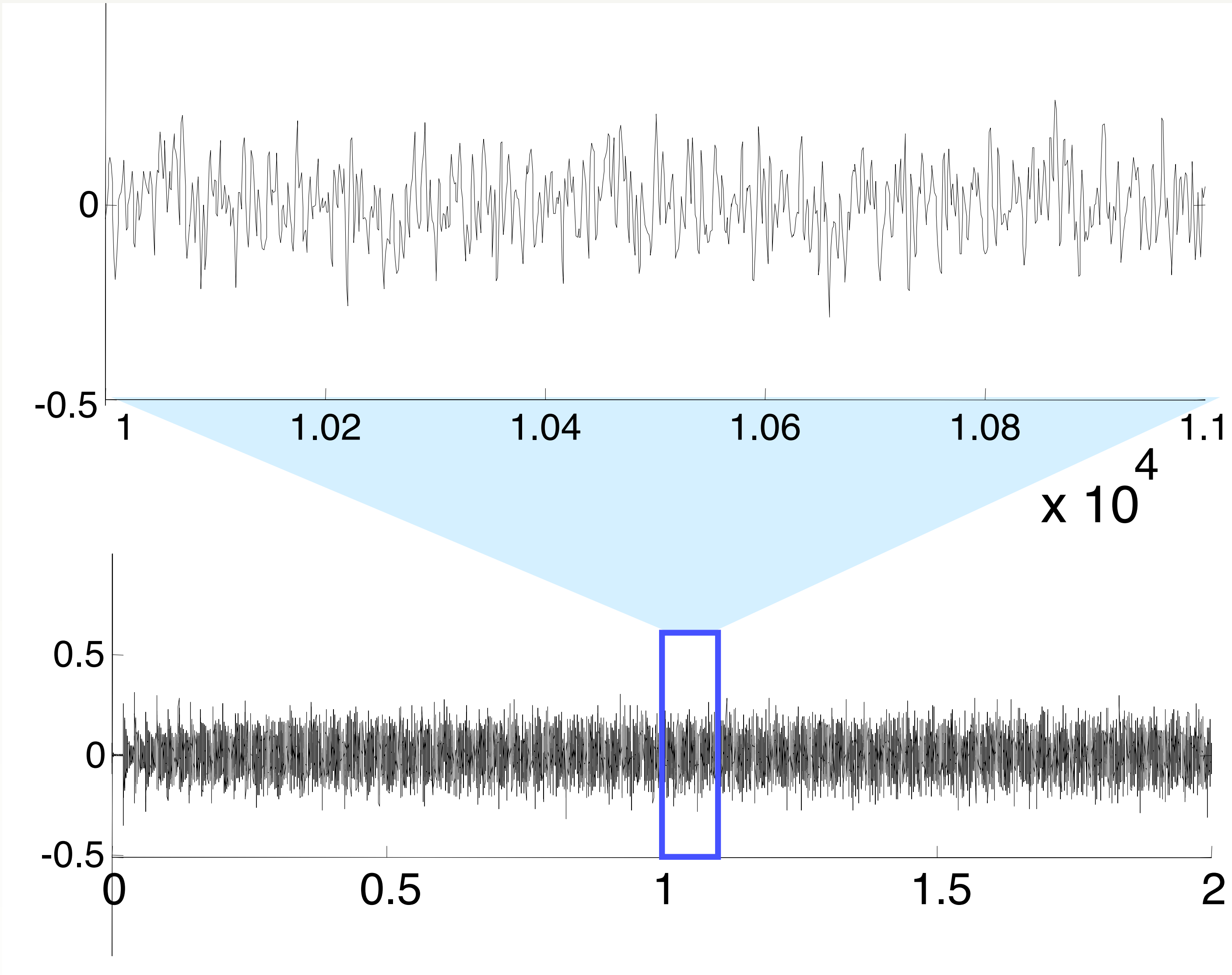
The term “nonlinear phenomenon” has no mathematical meaning!

"Chaos" in Linear Systems





“Chaos” in Linear Systems (cont.)



has continuous spectrum (PSD) over long, but finite times

“Chaos” in Linear Systems (cont.)

- With similar schemes,

  can generate stochastic processes with any prescribed PSD

- Clearly not the most efficient method to generate such processes!

Moral: *Sometimes a low dimensional nonlinear model is easier to handle than a high dimensional linear one*

Another example: *Optimal filtering equations*

Phrases to avoid

A well defined mathematical object: **nonlinear mapping**

mapping between two vector spaces that does not preserve their linear structure

Avoid:

- Nonlinear phenomenon (Science, Behavior, etc...)
- A linear system can not do this
- This is a nonlinear effect
- Even the term “nonlinear system” is a little ambiguous

To make sure you are on firm ground:

use the term nonlinear only in its original mathematical meaning

The End