Structured Stochastic Uncertainty

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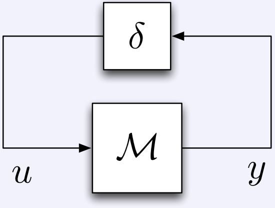
Stochastic Perturbations



- Perturbations are iid gains: $u(k) = \delta(k)\, y(k)$
- $\mathcal{M}$ is LTI
- nec & suff condition is the H^2 norm $\|\mathcal{M}\|_2^2 < \frac{1}{\sigma_\delta}$

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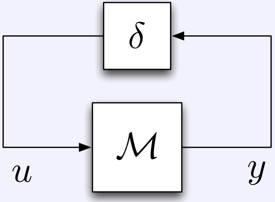

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- $\mathcal{M}$ is LTI
- nec & suff condition is the H^2 norm $\|\mathcal{M}\|_2^2 \;<\; \frac{1}{\sigma_\delta}$
- LMI proof (S. Boyd, '84)

$x(k+1) \;=\; (A \;+\; BC\,\delta(k))\;\, x(k)$ Linear System w Mult. Noise

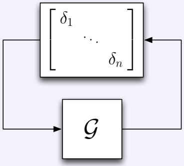
$P(k) := \mathcal{E}\left\{x(k)x(k)^T\right\}$

$$\Rightarrow \qquad P(k+1) \;=\; A\, P(k)\, A^T \;+\; \sigma_\delta BC\, P(k)\, C^T B^T$$

LMI cond. for $P(K) \overset{k\to\infty}{\to} 0$ $\quad = \quad$ LMI cond. for $\|\mathcal{M}\|_2^2 \;<\; \frac{1}{\sigma_\delta}$

Structured Stochastic Perturbations

Perturbations $\delta_1(t), \ldots, \delta_n(t)$ are iid



nec & suff cond. for Mean Square Stability given by *matrix of H^2 norms* !

$$\rho\left(\begin{bmatrix} \|g_{11}\|_2^2 & \cdots & \|g_{1n}\|_2^2 \\ \vdots & \ddots & \vdots \\ \|g_{n1}\|_2^2 & \cdots & \|g_{nn}\|_2^2 \end{bmatrix}\right) < \frac{1}{\sigma_\delta}$$

(Hinrichsen&Pritchard ’95, Lu&Skelton ’02, Elia ’04)

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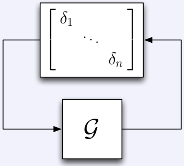

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- Proof involves LMIs and scalings

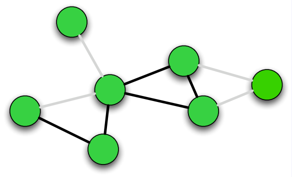
  cf. time-varying L^2 and L^∞-norm bounded perturbations
- Not clear how to generalize to correlated δs

Applications of Structured Stochastic Perturbations

- Network dynamics with link/node failures, etc.

(Elia, Patterson & Bamieh)

$$x(t+1) = A\,x(t) \;+\; \left(\sum_{i=1}^{M} \mu_i(t)\; b_i b_i^*\right)\; x(t)$$



Linear system w. multiplicative noise

Applications of Structured Stochastic Perturbations

- Network dynamics with link/node failures, etc.

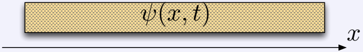
  (Elia, Patterson & Bamieh)
  (“discrete space”)

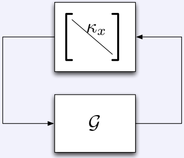
- PDEs with random coefficients (“continuous space”)

$$\frac{\partial}{\partial t}\psi(x,t) \;=\; \left(\mathcal{A} \;+\; \mathcal{B}\,\delta(x,t)\,\mathcal{C}\right)\;\psi(x,t)$$

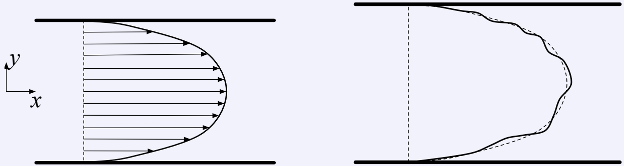
  - e.g. problems from random materials

$$\frac{\partial}{\partial t}\psi(x,t) \;=\; \left(\bar{\kappa} \;+\; \kappa(x,t)\right)\frac{\partial^2}{\partial x^2}\psi(x,t)$$





Stochastic Hydrodynamic Stability and Turbulence



Uncertain base flow

$$U(x,y,t) \;=\; \bar{U}(y,t) \;+\; \delta(x,y,t)$$

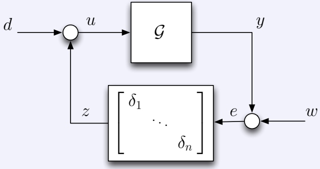
spatiotemporal correlations of δ should be “dialed into” the model

An Input-Output Approach to Structured Stochastic Perturbations

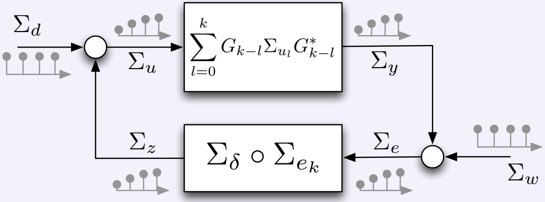
An IO Approach to Stochastic Stability

Look at the dynamics of the *correlation matrix sequences* Σ_{u_k}, etc.

stochastic vector signals



deterministic, matrix-valued signals



$$\Sigma_{\boldsymbol{\delta}} := \mathcal{E}\left\{ \begin{bmatrix} \delta_1 \\ \vdots \\ \delta_n \end{bmatrix} \begin{bmatrix} \delta_1 & \cdots & \delta_n \end{bmatrix} \right\}$$

↑
monotone system

- The "loop gain" operator plays a central role

$$\mathcal{L}(X) := \Sigma_{\boldsymbol{\delta}} \circ \left(\sum_{l=0}^{\infty} G_l \, X \, G_l^* \right)$$

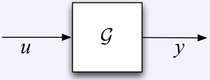
complexity of $\mathcal{L}$ scales w # of perturbations, *not state space dimension*

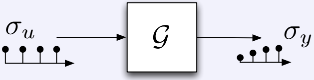
B. BAMIEH, *Structured stochastic uncertainty*, 2012 50th Annual Allerton Conference.

IO notion of Mean Square Stability

- **Def:** $\mathcal{G}$ is *Mean-Square Stable* (MSS) if for white input process u , output process y has uniformly bounded variance

$$y_k = \sum_l g_{k-l}\, u_l \qquad\qquad \sigma_{y_k} = \sum_l g_{k-l}^2\, \sigma_l$$





$$\sigma_{y_k} := \mathcal{E}\{y_k^* y_k\} \leq \underbrace{\left(\sum_k g_k^2\right)}_{=\ \|\mathcal{G}\|_2^2} \underbrace{\left(\sup_k \sigma_{u_k}\right)}_{\|\sigma_u\|_\infty} \qquad k \in \mathbb{Z}^+$$

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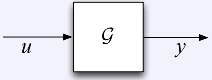
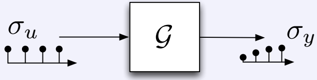
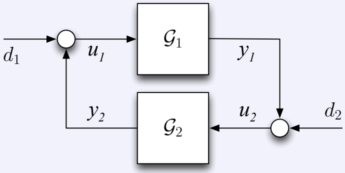
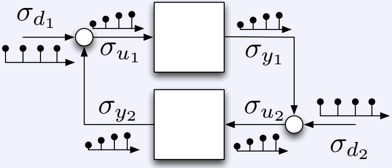




- MSS of feedback systems (insert disturbances, d_1, d_2 white)

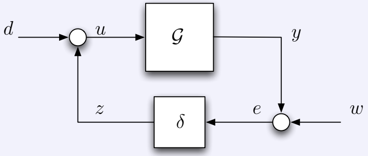




MSS Feedback Stability

$\Leftrightarrow$ All internal signals have uniformly bounded variance sequences

nec & suff Small Gain Condition (SISO)



- δ iid $\Rightarrow$ z is white (δ "whiten's" e)

$$\Rightarrow \quad \sigma_{z_k} = \sigma_\delta \, \sigma_{e_k}$$

- u is also white $\Rightarrow \quad \sigma_{y_k} \leq \|\mathcal{G}\|_2^2 \, \sigma_{u_k}$
- y is colored, but uncorrelated with w $\Rightarrow \quad \sigma_{e_k} = \sigma_{y_k} + \sigma_w$
- go around the loop with the *variance sequences*

$$\left(1 - \sigma_\delta \|\mathcal{G}\|_2^2\right) \; \|\sigma_u\|_\infty \;\leq\; \sigma_\delta \sigma_w \,+\, \sigma_d \qquad \text{suff}$$

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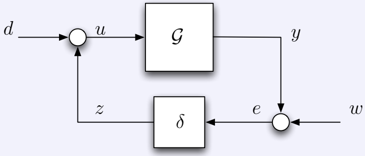

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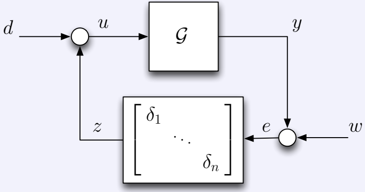
$$\left(1 - \sigma_\delta \|\mathcal{G}\|_2^2\right) \|\sigma_u\|_\infty \leq \sigma_\delta \sigma_w + \sigma_d \qquad \text{suff}$$

$$\sigma_\delta \|\mathcal{G}\|_2^2 \geq 1 \Rightarrow \sigma_{u_k} \text{ unbounded as } k \longrightarrow \infty \qquad \text{nec}$$

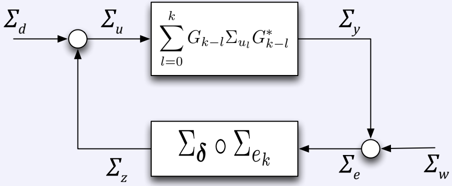
don't need to construct "destabilizing" δs!

nec & suff Structured Small Gain Condition

Look at the dynamics of the *correlation matrix sequences* Σ_{u_k}, etc.



$\leftrightarrow$



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$\uparrow$

monotone system

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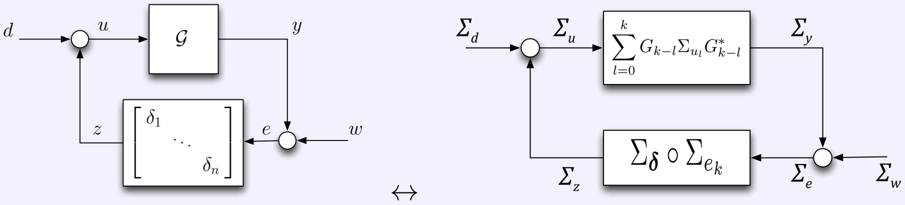

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↑

monotone system

- going around the loop

$$\Big(I \;-\; \mathcal{L}\Big)(\Sigma_{u_k}) \;\leq\; \Sigma_d \;+\; \Sigma_{\boldsymbol{\delta}} \circ \Sigma_w$$

$$\mathcal{L}(X) \;:=\; \Sigma_{\boldsymbol{\delta}} \circ \left(\sum_{l=0}^{\infty} G_l \, X \, G_l^* \right) \qquad \longleftarrow \quad \text{“loop gain”}$$

Properties of the "Loop Operator" $\mathcal{L}$

$$\mathcal{L}(X) \;:=\; \Sigma_{\delta} \circ \left(\sum_{l=0}^{\infty} G_l \, X \, G_l^* \right)$$

- $\mathcal{L}$ maps pos. s. def. matrices to pos. s. def. matrices
- It is thus "cone invariant" for the cone of pos. s. def. matrices
  & $\exists$ a Perron eigenvalue and corresponding pos. s def. *eigenmatrix*

(Parrilo & Khatri '00)

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these properties imply

$$(1 - \rho(\mathcal{L})) \; \Sigma_{u_k} \leq \left(I - \mathcal{L} \right) (\Sigma_{u_k}) \;\leq\; \Sigma_d \;+\; \Sigma_{\delta} \circ \Sigma_w$$

MSS stability condition (nec & suff)

$$\rho\,(\mathcal{L}) \;<\; 1$$

Special Case of iid δs

- δs iid $\quad \rightarrow \quad \Sigma_{\boldsymbol{\delta}} = I$
-

$$\mathcal{L}(X) \;:=\; I \circ \left(\sum_{l=0}^{\infty} G_l \, X \, G_l^* \right) \;=\; \text{diag} \left(\sum_{l=0}^{\infty} G_l \, X \, G_l^* \right)$$

- The "eigen-matrices" of $\mathcal{L}$ must be diagonal matrices !

$$\mathcal{L}(X) \;=\; \lambda X$$

- How does $\mathcal{L}$ act on diagonal matrices?

$$\begin{bmatrix} w_1 & & \\ & \ddots & \\ & & w_n \end{bmatrix} = \mathcal{L}\left(\begin{bmatrix} v_1 & & \\ & \ddots & \\ & & v_n \end{bmatrix} \right) \;\Leftrightarrow\; \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix} = \begin{bmatrix} \|g_{11}\|_2^2 & \cdots & \|g_{1n}\|_2^2 \\ \vdots & \ddots & \vdots \\ \|g_{n1}\|_2^2 & \cdots & \|g_{nn}\|_2^2 \end{bmatrix} \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$$

Therefore

$$\text{eigs}(\mathcal{L}) \;=\; \text{eigs}\left(\left[\|g_{ij}\|_2^2 \right] \right)$$

Mutually Correlated δs (but temporally white)

$$\mathcal{L}(X) \;:=\; \Sigma_{\boldsymbol{\delta}} \circ \left(\sum_{l=0}^{\infty} G_l \; X \; G_l^* \right)$$

- $\mathcal{L} : \mathbb{R}^{n \times n} \longrightarrow \mathbb{R}^{n \times n}$

  In general, it involves terms like

$$\sum_{k=0}^{\infty} g_{ij}(k) \; g_{lm}(k)$$

  inner products between subsystems' impulse responses

- At worst: $\mathcal{L}$ represented as an $n^2 \times n^2$ matrix

Further Work

- Robust Performance and Correlations
- Spatial correlations in δs and $\mathcal{G}$ have special structure
  e.g. spatial invariance $\Rightarrow$ simpler conditions
  useful for applications to large-scale systems

- Partial Differential Equations
  $\mathcal{L}$ is a map on pos. s. spatial operators

- Temporal correlations in δs ??? probably involves other aggregates of the impulse response sequence $\{g_k\}$