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OPTIMAL DISTRIBUTED CONTROL WITH DISTRIBUTED DELAYED MEASUREMENTS

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Abstract: We consider the problem of optimal distributed controller design of semi-decentralized controllers for a special class of spatially distributed systems. This class includes spatially invariant and distributed systems with an inherent temporal delay in the interaction of distant sites. We consider the problem of optimal design of distributed controllers where controller information passing is as fast or faster than the plant. We show how the YJBK parametrization of such stabilizing controllers yields a convex parametrization for this class.

1. INTRODUCTION

We consider spatially distributed systems where all signals are functions of both spatial and temporal variables. The theory of such spatio-temporal systems has been worked out in some detail. We consider only spatially distributed systems with the additional property that the dynamics are spatially invariant. For recent work on this class and some of the background for the present work, we refer the reader to Bamieh et al. (2001); Paganini and Bamieh (1998) and the references therein.

One of the major issues in the design of such distributed controllers is the communications requirements between individual controller sub-systems. One of the applications of this design methodology is to design controllers for large arrays of Micro-Electro-Mechanical System (MEMS), in which there are potentially tens of thousands of actuator/sensor and imbeded control sub-systems. For systems of this size and configuration, centralized controllers are not an option. It turns out that optimally designed centralized controllers have an inherent localization property which enables them to be implemented using distributed control elements with limited communication requirements Paganini and Bamieh (1998). Several researchers have recently been looking at the problem of explicity imposing constraints on communication requirements between controller subsystems. Among these are approcahes based on LMI's and convex optimization techniques (see Ayres and Paganini (2000); D'Andrea (June 1998); Dullerud et al. (December 1998) and the references therein).

In this paper we show how the problem where the controller is constrained such that information is propagated with a delay dependent on distance between sub-systems (a structure we refer to as "funnel"-causality), can be solved optimally as a convex problem provided that the plant has a similar funnel-causality structure. Before stating this problem precisely we present some material on spatio-temporal causality.

2. SPATIO-TEMPORAL CAUSALITY

Our description of distributed systems will be in terms of their spatio-temporal impulse and frequency responses. A signal $u(x,t)$ is a function of a spatial variable x and a temporal variable t. In what follows, x and t can be either discrete or continuous. In the solution to the $\mathcal{H}_2$ problem later, we restrict ourselves to discrete temporal and spatial variables.

Two signals, u and y are related by a spatially-invariant distributed system if we can write

$$y(x,t) \;=\; \int\int h(x-\xi, t-\tau)\; u(\xi,\tau)\; d\tau\; d\xi, \quad (1)$$

where h is the spatio-temporal impulse response. For uniformity of notation, we use this convolution integral to denote sums as well in the case when $d\tau$ and $d\xi$ are discrete measures on the set $\mathbb{Z}$. We will restrict attention to the class of temporaly causal impulse responses that have the following property,

$$\sup_{t\in[0,T]} \int_{-\infty}^{\infty} |h(x,t)|\; dx < \infty,$$

for any $T>0$. This can be understood as requiring h to be in L^1 in the spatial coordinate, and in L_e^∞ (the extended L^∞ space) in the temporal coordinate. This class is large enough to contain most temporaly causal (but not necessarily stable) spatio-temporal systems. The above bound allows for composition and inversion of such systems.

The spatio-temporal impulse response $h(x,t)$ can be visualized as a function in the plane (x,t). Temporal causality of h is equivalent to the requirement that $h(x,t)=0$ in the half plane $t \le 0$. Physical systems have temporal causality, but not necessarily spatial causality. As opposed to purely temporal systems, where only one notion of causality is natural, there are many possible notions of causality for spatio-temporal systems. Systems that have a constant finite propagation speed (e.g. the wave equation) are such that h has its support in a "light cone", i.e.

$$h(x,t)=0, \qquad \text{for } t < \gamma x$$

(see figure 1.a), where $1/\gamma$ is the speed of propagation. This type of causality maybe referred to as "cone-causality". We will be considering systems who's impulse response has support in slightly more general domains.

Definition 1. A scalar valued function $f(x)$ is said to be a *propagation function* if f is non-negative, $f(0)=0$ and such that $\{f(x),\ x\ge 0\}$ and $\{f(x),\ x\le 0\}$ are concave functions respectively.

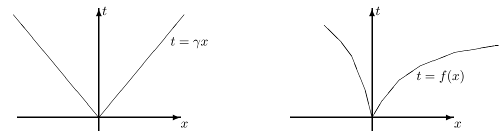

Fig. 1. (a) Cone causality (b) Funnel causality

Definition 2. A system is said to have the property of *funnel-causality* if its impulse response is such that

$$h(x,t)=0, \qquad \text{for } t < f(x),$$

where $f(x)$ is a propagation function. In other words, if its impulse response is supported in a funnel shaped region (see figure 1.b).

Intuitively, any effect in a funnel-causal systems takes at least $f(x)$ time units to travel a distance x. The reason for restricting the propagation function c to have concave segments is that such a class of systems turns out to be closed under convolutions. This latter property will be useful in establishing the convexity of controller design problems later on.

We now state a result on the composition of two funnel-causal systems. If h is a spatio-temporal system, we use the same symbol h to refer to its spatio-temporal impulse response (the function $h(x,t)$), and we use the expression $\mathrm{supp}\,(h)$ to refer to the region in the (x,t) plane where $h(x,t)$ is supported. If f is a propagation function, we denote by S_f the set

$$S_f \;:=\; \{(x,t);\ t \ge f(x)\}\,.$$

S_f is the set "above" the curve c in figure 1.b.

Lemma 1. Let h_1 and h_2 be two funnel-causal systems such that

$$\mathrm{supp}\,(h_1) \;\subset\; S_f, \qquad \mathrm{supp}\,(h_2) \;\subset\; S_f,$$

where f is some propagation function. Then the composition $h_3 = h_1 * h_2$ is such that

$$\mathrm{supp}\,(h_3) \;\subset\; S_f.$$

The preceding lemma characterizes an important property of funnel-causal systems. The composition of two such systems is also a funnel-causal system where effects propagate as fast as the fastest of the two systems. To make this precise, let h_1 and h_2 be systems whose support is such that $\mathrm{supp}\,(h_1) \subset$

S_{f_1} and $\text{supp}\,(h_2) \subset S_{f_2}$, where $S_{f_1} \subset S_{f_2}$. This means that effects in h_2 propagate faster than in h_1. Lemma 1 then implies that $\text{supp}\,(h_1 * h_2) \subset S_{f_2}$.

Lemma 1 implies that for a given propagation function, the class of funnel-causal systems is closed under compositions. It is a trivial fact that this class is closed under additions as well. Furthermore, it can also be shown that this class is closed under inversions. Taking all three properties together, i.e. closure under additions, compositions and inversions, we conclude that the class of funnel-causal systems is closed under general linear fractional transformations with coefficients that are themselves funnel-causal. This then implies that the YJBK parameterization can be used to nicely parameterize all funnel-causal stabilizing controllers.

3. PROBLEM DEFINITION AND YJBK PARAMETRIZATION

In the design of distributed controllers for spatio-temporal systems it is often desired to impose some decentralized structure on the controller. A fully centralized controller is often impractical in large scale systems though it has the best performance. Explicit decentralization is a notoriously difficult control problem. Perhaps an indication of the difficulty of this problem is that the set of all achievable closed loop maps with decentralized control is not in general a convex set. However, we will now consider controllers with prescribed funnel-causality which yield convex closed loops for certain plants.

Let a propagation function f be given, and denote by L_f the set of all linear spatially invariant systems with impulse responses that have support in S_f. Consider the standard configuration for disturbance attenuation in Figure 2 where the plant G and the controller K are spatially and temporally invariant systems. A central observation in this paper is that if $\text{supp}\,(g_{22}) \subset S_f$ for some propagation function c, then the problem of designing controllers with support in S_f is convex.

Problem definition: Consider the standard problem in figure 2 with G_{22} such that $\text{supp}\,(g_{22}) \subset S_f$ for some propagation function f. Find the optimal feedback controller with the same funnel-causality constraint as G_{22}, i.e.

$$\inf_{\substack{K \text{ stabilizing} \\ K \in L_f}} \|\mathcal{F}(G;K)\|, \qquad (2)$$

where the norm in question is any norm defined on spatio-temporal systems.

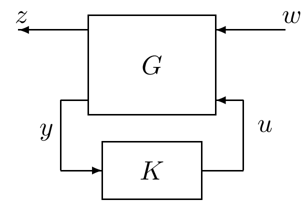

Fig. 2. The Standard Problem

Convexity of the set of closed loops

In order to employ the YJBK paramterization for the case of an unstable G_{22} we will assume the existence of a co-prime factorization. We will assume that we can factor $G_{22} = NM^{-1}$, and that there exists X and Y that solve the Bezout identity $XM - YN = I$, where N, M, X, Y are stable spatio-temporal systems. The next results shows that for a funnel-causal system, if a co-prime factorization can be found with funnel-causal factors, then the decentralization constraints on K transform to convex constraints on the Youla parameter Q, which in turn produces a convex set of achievable closed loop maps. In the next section we explain how we can find Bezout identity factors that are funnel causal for a class of spatio-temporal systems.

Proposition 1. Let $G_{22} \in L_f$ for some propagation function f. Let $G_{22} = NM^{-1}$ and $XM - YN = I$ with $N, M, X, Y \in L_f$ and stable. Then all stabilizing controllers K such that $K \in L_f$ are given by

$$K = (Y + MQ)(X + NQ)^{-1}, \qquad (3)$$

where Q is a stable system in L_f.

With the above parametrization, problem (2) becomes

$$\inf_{\substack{Q \text{ stable} \\ Q \in L_f}} \|H - UQV\|, \qquad (4)$$

where H, U, V are stable maps that depend only on G. Now since the set of stable $Q \in L_f$ is a linear subspace, and the mapping $Q \mapsto (H - UQV)$ is linear affine, then problem (4) is a convex problem, in particular, it is a minimum distance to a subspace problem. The difficulty of such a problem and whether it is finite or infinite-dimensional will depend on the norm used and the nature of the set

L_f (equivalently, the type of propagation function f).

4. FACTORIZATIONS

Co-prime factorizations and Bezout identities can be developed for spatio-temporal systems in a very similar manner to those for finite dimensional systems. Our interest however is to ensure that the factors and corresponding Bezout identity elements satisfy the same funnel-causality constraints as the plant. Rather than develop the most general procedure for doing co-prime factorization, we concentrate on the special case of plants where decentralized proportional feedbacks can be used to obtain stablizing state feedback and observer gains. This covers a large class of spatio-temporal systems derived from physical partial differential equations.

To begin with, let the input-output distributed system $y = Gu$ be given by a state space realization

$$\begin{aligned} \partial_t \psi &= A\psi \ + \ Bu \\ y &= C\psi \end{aligned}, \tag{5}$$

where ψ, u, and y are spatio-temporal signals, and A, B, C are translation invariant operators. We refer the reader to Bamieh et al. (2001) for the background and some of the results we later use related to such systems.

We now illustrate how to find co-prime factorizations and solve Bezout identities for such systems. The procedure is very similar to the finite dimensional case. The Bezout identity is Zhou et al. (1996)

$$XM - YN \ = \ I,$$

where $G = NM^{-1}$, and N, M, X and Y are stable systems. State space realizations for elements of the Bezout identity are given by

$$\begin{aligned} \begin{bmatrix} X & -Y \end{bmatrix} &= \left[\begin{array}{c|cc} A+LC & -B & L \\ \hline K & I & 0 \end{array}\right], \\ \begin{bmatrix} M \\ N \end{bmatrix} &= \left[\begin{array}{c|c} A+BK & B \\ \hline K & I \\ C & 0 \end{array}\right], \end{aligned} \tag{6}$$

where the spatial operators K and L are chosen such that $A+BK$ and $A+LC$ generate stable evolutions.

The difficulty with obtaining good co-prime factorizations for the problem of funnel-causality is that even if the original system is funnel-causal, the feedback gains K and L used to form the Bezout identity may destroy this property. We present below a criterion which avoids this problem when simple proportional gains K and L are used.

Proposition 2. Let a spatio-temporal system be given by the state space model (5) such that the impulse responses $e^{tA}B$, Ce^{tA} and $Ce^{tA}B$ are funnel-causal. If there exists proportional gains K and L (i.e. decentralized feedbacks) such that $A+BK$ and $A+LC$ are stable, then all elements of the Bezout identity (6) are funnel-causal.

The result above can be easily generalized to the case when the gains K and L are local spatial operators (e.g. spatial derivatives of any order), but we will not need this generality here. Although it is restrictive to assume that one can find stabilizing decentralized state feedbacks and observer gains, this property seems to hold for a large class of spatio-temporal systems with distributed control. This is illustrated with several examples in later sections. We note that for vector-valued input and output signals, a non-commutative version of lemma 1 can be stated. This has the standard form Vidyasagar (1995), and we do not repeat the formulae here.

Example: The wave equation

We illustrate the forgoing ideas using the wave equation. The partial differential equation

$$\partial_t^2 \psi(x,t) \ = \ c^2 \partial_x^2 \psi(x,t) \ + \ u(x,t), \tag{7}$$

is the standard wave equation with a distributed input. Its transfer function is given by $G(s,k) = \frac{1}{s^2+c^2k^2}$. This system can not be stabilized by proportional decentralized output feedback alone. A realization of this system has the form (5) as

$$\begin{aligned} \partial_t \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} &= \begin{bmatrix} 0 & I \\ c^2\partial_x^2 & 0 \end{bmatrix} \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} + \begin{bmatrix} 0 \\ I \end{bmatrix} u \\ \psi &= \begin{bmatrix} I & 0 \end{bmatrix} \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix}. \end{aligned}$$

Following Bamieh et al. (2001), this system can be analyzed by taking a Fourier transform in the spatial variables. Denoting the spatial Fourier variable by k (the wave number), the Fourier representation of the above system is

$$\begin{aligned} \frac{d}{dt} \begin{bmatrix} \psi_1(k,t) \\ \psi_2(k,t) \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ -c^2k^2 & 0 \end{bmatrix} \begin{bmatrix} \psi_1(k,t) \\ \psi_2(k,t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k,t) \\ \psi(k,t) &= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} \psi_1(k,t) \\ \psi_2(k,t) \end{bmatrix}, \end{aligned}$$

where for simplicity of notation we use the same symbol to denote a signal $\psi(x,t)$ and its spatial Fourier transform $\psi(k,t)$. To see that the system (7) has funnel causality, we compute e^{tA}. Note that the 2×2 matrix A can be diagonalized by

$$\begin{bmatrix} 0 & 1 \\ -c^2k^2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ ick & -ick \end{bmatrix} \begin{bmatrix} ick & 0 \\ 0 & -ick \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{-i}{2ck} \\ \frac{1}{2} & \frac{i}{2ck} \end{bmatrix},$$

with $c>0$. This diagonalization then implies that

$$\begin{aligned}
&\exp\left\{ t \begin{bmatrix} 0 & 1 \\ -c^2k^2 & 0 \end{bmatrix} \right\} = \\
&= \begin{bmatrix} 1 & 1 \\ ick & -ick \end{bmatrix} \begin{bmatrix} e^{ikct} & 0 \\ 0 & e^{-ikct} \end{bmatrix} \begin{bmatrix} 1/2 & -i/2ck \\ 1/2 & i/2ck \end{bmatrix} \\
&= \frac{1}{2} \begin{bmatrix} e^{ikct} + e^{-ikct} & \frac{1}{ick}\left(e^{ikct} - e^{-ikct}\right) \\ ick\left(e^{ikct} - e^{-ikct}\right) & e^{ikct} + e^{-ikct} \end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{2}\left(e^{ikct} + e^{-ikct}\right) & t\ \mathrm{sinc}(kct) \\ -c^2k^2\ t\ \mathrm{sinc}(kct) & \frac{1}{2}\left(e^{ikct} + e^{-ikct}\right) \end{bmatrix}
\end{aligned}$$

As is well known, the symbol e^{-ickt} is the Fourier representation of the operator T_{ct} of right translation by distance ct. Multiplication by $t\,\mathrm{sinc}(kct)$ represents convolution with the "rectangular" function $\frac{1}{2c}\mathrm{rec}(\frac{1}{ct}x)$, where [1]

$$\mathrm{rec}(x) := \begin{cases} 1 & |x| \le 1 \\ 0 & |x| > 1 \end{cases}.$$

If we denote by R_{ct} the operation of spatial convolution with $\mathrm{rec}(\frac{1}{ct}x)$, then we can represent e^{tA} as

$$e^{tA} = \frac{1}{2} \begin{bmatrix} T_{ct} + T_{-ct} & \frac{1}{c}R_{ct} \\ c\partial_x^2\, R_{ct} & T_{ct} + T_{-ct} \end{bmatrix}.$$

Now, ∂_x^2 is a local operator, while T_{ct}, T_{-ct} and R_{ct} are non-local. However, they are all funnel causal with propagation function $f(x) = \frac{1}{c}t$ (i.e. they are cone causal). To see this, note that their respective impulse responses are

$$\begin{aligned}
&(T_{ct})(x,t) = \delta(x-ct), \quad (T_{-ct})(x,t) = \delta(x+ct), \\
&(R_{ct})(x,t) = \mathrm{rec}(\frac{1}{ct}x),
\end{aligned}$$

all of which are supported in the region $\{(x,t);\ ct > x\}$.

[1] This is obtained from the Fourier transform pair: $\mathrm{rec}(\frac{1}{\alpha}x) \leftrightarrow 2\alpha\ \mathrm{sinc}(\alpha k)$

We have thus established that all elements of e^{tA} are funnel-causal. Since B and C are constants, this system satisfies the first set of assumptions of proposition 2. We now show how to easily find stabilizing proportional state feedback and observer gains. First, to find a suitable state feedback gain K

$$\begin{aligned}
A + BK &= \begin{bmatrix} 0 & 1 \\ -c^2k^2 & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} k_1 & k_2 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 1 \\ -c^2k^2 + k_1 & k_2 \end{bmatrix}.
\end{aligned}$$

We set $k_1 = 0$. Then, the eigenvalues of $A + BK$ for each wave-number k are given by $k_2 \pm \frac{1}{2}\sqrt{k_2^2 - 4c^2k^2}$. Thus for $k_2 < 0$, the spectrum of the operator $A+BK$ is the set $\left[\frac{3}{2}k_2, \frac{1}{2}k_2\right] \bigcup (k_2 + j\mathbb{R})$, which has negative real part if $k_2 < 0$. Similarly, to find the observer gain, note that

$$A + LC = \begin{bmatrix} l_1 & 1 \\ -c^2k^2 + l_2 & 0 \end{bmatrix}.$$

Setting $l_2 = 0$, we find that the spectrum of $A + LC$ has negative real part if $l_1 < 0$. Choosing $l_1 = k_2 = -1$, we obtain stabilizing gains

$$K = \begin{bmatrix} 0 & -1 \end{bmatrix}, \quad L = \begin{bmatrix} -1 \\ 0 \end{bmatrix}.$$

Now we compute the co-prime factors using formulae (6)

$$\begin{bmatrix} X & -Y \end{bmatrix} = \left[\begin{array}{cc|cc} -1 & 1 & 0 & -1 \\ -c^2k^2 & 0 & -1 & 0 \\ \hline 0 & -1 & 1 & 0 \end{array}\right],$$

$$\begin{bmatrix} M \\ N \end{bmatrix} = \left[\begin{array}{cc|c} 0 & 1 & 0 \\ -c^2k^2 & -1 & 1 \\ \hline 0 & -1 & 1 \\ 1 & 0 & 0 \end{array}\right].$$

We compute the factors to be

$$M = \frac{s^2 + c^2k^2}{s^2 + s + c^2k^2}, \qquad X = \frac{s^2 + 2s + c^2k^2 + 1}{s^2 + s + c^2k^2},$$

$$N = \frac{1}{s^2 + s + c^2k^2}, \qquad -Y = \frac{-c^2k^2}{s^2 + s + c^2k^2}.$$

The funnel-casuality of all the above factors is guaranteed by proposition 2.

A closed loop mapping such as sensitivity can then be written in terms of the Q parameter as

$$(I + GK)^{-1} = XM + NM\,Q.$$

Inner-outer factorizations

In linear quadratic minimum distance problems such as $\mathcal{H}_2$ and $\mathcal{H}_\infty$, inner-outer factorizations play an important role. We now illustrate how to perform inner-outer factorizations for spatio-temporal systems. The following is applicable to spatio-temporal system with discrete or continuous spatial and temporal variables, however for simplicity we will present the case of discrete time and discrete space.

Certain stable spatio-temporal systems can be considered as mappings on L^2. If a system H can be described in terms of an impulse response like (1), it is called L^2-stable if it is a bounded linear mapping on $L^2(\mathbb{Z}^2)$. A combined temporal Laplace transform and spatial Fourier transform gives a transfer function description of the system (1) by

$$\begin{aligned} y(k,t) &= \sum_{-\infty<l<\infty} \sum_{0\leq i<\infty} h(k-l,t-i)\ u(l,i) \\ &\Updownarrow \\ Y(\theta,\lambda) &= H(\theta,\lambda)U(\theta,\lambda), \end{aligned}$$

where the combined transform of any signal is defined by

$$\begin{aligned} Y(\theta,\lambda) &:= \sum_k \sum_t y(k,t)\ \lambda^t \left(e^{j\theta}\right)^k \\ &= \sum_k \sum_t y(k,t)\ \lambda^t z^k \Big|_{z=e^{j\theta}}, \end{aligned}$$

where $\theta \in [0,2\pi]$, and $\lambda \in \mathbb{C}$ in the appropriate region of convergence of the transform. Note that λ is the temporal transform variable, while z is the spatial transform variable. This definition emphasizes the fact that there is no spatial causality structure in such systems, and only the value of the transform at $z = e^{j\theta}$ is important.

A spatially invariant system H which is L^2 stable is said to have an inner-outer factorization if it can be decomposed as

$$H \ = \ H_i\ H_o,$$

where H_i is an isometry on L^2, and H_o is causally invertible on L^2. Now, given a transfer function $H(\theta,\lambda)$ of a spatio-temporal system, an inner-outer factorization of H can be obtained from the inner-outer factorization at each θ. In other words, for every $\theta \in [0,2\pi]$ we can decompose

$$H(\theta,\lambda) \ = \ H_i(\theta,\lambda)\ H_o(\theta,\lambda),$$

where for each θ, the above is an inner-outer factorization of the temporal system $H(\theta,.)$. Now, it is a consequence of Parseval's equality that the spatio-temporal system $\{H_i(\theta,\lambda)\}$ is an isometry, and that the system $\{H_o(\theta,\lambda)\}$ is causally invertible.

As in the temporal case, the isometry property can be recast as

$$H^\sim(\theta,\lambda)\ H(\theta,\lambda) \ = \ I,$$

where

$$H^\sim(\theta,\lambda) \ := \ H^*(\theta,\frac{1}{\lambda}),$$

and $H^\sim$ is also an isometry, though not necessarily a causal system.

5. CONCLUSION

We considered optimal feedback control for spatially invariant distributed systems with an inherent temporal delay in the interaction of neighbouring sites. The type of delay structure that leads to convex optimal control problems has been characterized using the concept of funnel causality. We have shown how to constructively convert such optimal feedback control problems to model matching problems using special versions of the YJBK parameterization.

Applications of this theory to specific $\mathcal{H}_2$ problems will be presented in the final version of this paper.

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